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# MECHANICS VIA THE CALCULUS





# MECHANICS

VIA THE

# CALCULUS

*By*

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## PREFACE TO THIRD EDITION

THE present edition of this book has given the authors an opportunity not only to make a number of minor corrections and improvements, but to include further material on Principal Axes of Inertia and on Instantaneous Centre of Rotation.

In view of the interest taken in the subject of Vector Analysis in recent years, a chapter has also been added to introduce this subject to the student, who, it is hoped, will be persuaded thereby to pursue his studies further in that direction.

The large number of examples—always a popular feature of the book—has been supplemented by some additional problems of a type favoured by examiners in recent years. Apart from the worked examples in the text, the book now provides about 550 problems for solution.

The continuous steady demand which the work has enjoyed for so many years encourages the modest hope that the authors' aims have in some measure been achieved and that the book will continue to serve the cause of Mathematical teaching in Universities, Technical Colleges and Grammar Schools.

The authors are indebted to Mr. F. M. Goldner, M.A., and to Mr. A. E. Chapman, M.Sc., for their suggestions relating to the chapter on Vector Analysis.

*January, 1950*

P. W. N.  
W. S. L.

### PUBLISHER'S NOTE

THIS third impression of the Third Edition incorporates a number of amendments and corrections. The publishers acknowledge with thanks the further help of Mr. F. M. Goldner, and some valuable suggestions from Mr. A. R. Khalifa, M.Sc., Calc., of the University of Dacca.

*Spring, 1957*

## PREFACE TO FIRST EDITION

As the outcome of our experience in teaching Mechanics to Secondary, Technical and University students, we hope that this text-book will serve to bridge the gap which we feel exists between the elementary text-books on the subject and the complete treatises which usually assume an advanced knowledge of the Calculus.

The subject matter includes Statics, Dynamics of a Particle, Rigid Dynamics, and Hydrostatics ; moreover, as the title implies, the subject has been developed, so far as is consistent with the logical development of each branch, according to the difficulty of the Calculus involved. The first two chapters introduce the notation of the Calculus and deal with linear motion, but problems involving any differential equations other than those which can be solved by separation of the variables, are dealt with in a much later chapter. The applications of integration then lead to the discussion of Centres of Gravity and of Attractions. Chapters on Virtual Work, Moments of Inertia, and the Equilibrium of Strings then follow. There are also two chapters on the Motion of Rigid Bodies and two on Hydrostatics. The book concludes with a study of problems in Linear and two-Dimensional Motion, Central Orbits, etc., and with an introduction to the study of Bending Moments in Beams, and of Elasticity.

The book, it is hoped, will be of use to senior boys in Public and Secondary Schools, and to students working for University Degrees in the Faculties of Science and Engineering, the subject matter being sufficient, we trust, to cover the syllabuses in Applied Mathematics as set out for Pass Degrees at any University. Having some considerable acquaintance with the difficulties of students when first approaching this subject, we have incorporated many examples in the text ; in fact, it will be found that a large proportion of the subject matter is devoted to the solution of typical examples.

We have to thank Mr. F. J. Swan, B.A., for the assiduous care he has given to the reading of the proof sheets, and Mr. D. Humphrey, B.A., B.Sc., and Mr. B. E. Lawrence, B.A., B.Sc., for their kind assistance. We must also express our thanks to the Syndics of the Cambridge University Press and to the Senate of the London University for their kind permission to print their examination questions.

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## CHAPTER I

### LINEAR MOTION I

**1. The Velocity of a Point is the Rate of Change of its Position.**—The acceleration of a point is the rate of change of its velocity.

The velocity and acceleration of a point involve direction as well as magnitude and are thus vector quantities, that is they can be completely represented by straight lines. In Chapters I and II, problems in which the directions are constant are considered.

**2. General Expressions for Velocity and Acceleration.**—A particle moves along a straight line; O is a fixed point on the line and the particle travels from O to P, a distance  $s$ , in time  $t$ . After a further small interval of time  $\delta t$ , the particle is at Q where PQ equals  $\delta s$ . Then the average velocity from P to Q is  $\frac{\delta s}{\delta t}$ . When Q is at an infinitesimal small distance

from P, the velocity,  $v$ , at P is given by  $\text{Lt}_{\delta t \rightarrow 0} \frac{\delta s}{\delta t}$  i.e.  $v = \frac{ds}{dt}$ .

Suppose the velocity at Q is  $v + \delta v$ , then the average acceleration from P to Q is  $\frac{\delta v}{\delta t}$ : therefore the acceleration

at P is  $\text{Lt}_{\delta t \rightarrow 0} \frac{\delta v}{\delta t}$  i.e.  $\frac{dv}{dt}$  or  $\frac{d}{dt}\left(\frac{ds}{dt}\right)$  i.e.  $\frac{d^2s}{dt^2}$ .

As  $\frac{\delta v}{\delta t} = \frac{\delta v}{\delta s} \cdot \frac{\delta s}{\delta t}$ , on taking the limit when  $\delta t \rightarrow 0$ ,  $\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \cdot \frac{dv}{ds}$ . Thus the general expressions for the acceleration of a particle moving along a straight line are

$$\frac{dv}{dt}, \text{ or } \frac{d^2s}{dt^2} \text{ or } v \frac{dv}{ds}.$$

**3. Equations Determining Motion.**—Under specified initial conditions and when one relation between any two of the four quantities, Time ( $t$ ), Distance ( $s$ ), Velocity ( $v$ ) and Acceleration ( $a$ ), is given, the motion can be completely determined.

Suppose the initial conditions to be  $s = 0$  and  $v = u$  when  $t = 0$ , the six cases that arise are as follows:—

I. Given  $s = f(t)$ , to express  $v$  and  $a$  in terms of  $t$ .

$$v = \frac{ds}{dt} = f'(t).$$

and

$$a = \frac{d^2s}{dt^2} = f''(t).$$

II. Given  $v = f(t)$ , to express  $a$  and  $s$  in terms of  $t$ .

$$a = \frac{dv}{dt} = f'(t).$$

As

$$\frac{ds}{dt} = f(t),$$

 $\therefore$ 

$$\int ds = \int f(t) dt;$$

i.e.  $s = F(t) + C$  where  $\frac{d}{dt}F(t) = f(t)$  and  $C$  is a constant.

When

$$t = 0, s = 0,$$

 $\therefore$ 

$$0 = F(0) + C, \text{ so that } C = -F(0).$$

Substituting,  $s = F(t) - F(0)$ .III. Given  $a = f(t)$  to express  $v$  and  $s$  in terms of  $t$ .

As

$$\frac{dv}{dt} = f(t),$$

then

$$\int dv = \int f(t) dt,$$

i.e.  $v = F(t) + C$ , where  $\frac{d}{dt}F(t) = f(t)$ .

When

$$t = 0, v = u,$$

 $\therefore$ 

$$u = F(0) + C \text{ so that } C = u - F(0).$$

Substituting,

$$v = F(t) - F(0) + u.$$

Integrating with respect to  $t$ , we have

$$s = \int F(t) dt - t \cdot F(0) + ut + C_1,$$

where  $C_1$  is a constant which can be found from the condition that when  $t = 0, s = 0$ .IV. Given  $v = f(s)$ , to express  $a$  and  $t$  in terms of  $s$ .

$$\frac{ds}{dt} = f(s)$$

 $\therefore$ 

$$a = \frac{d^2s}{dt^2} = f'(s) \cdot \frac{ds}{dt} = f'(s) \cdot f(s).$$

Also

$$\int dt = \int \frac{ds}{f(s)},$$

 $\therefore$ 

$$t = F(s) + C, \text{ where } \frac{d}{ds}F(s) = \frac{1}{f(s)}.$$

When

$$t = 0, s = 0,$$

 $\therefore$ 

$$0 = F(0) + C.$$

V. Given  $a = f(s)$ , to express  $v$  and  $t$  in terms of  $s$ 

$$v \frac{dv}{ds} = f(s)$$

$$\int v dv = \int f(s) ds,$$

$$\text{i.e.} \quad \frac{v^2}{2} = F(s) + C, \text{ where } \frac{d}{ds} F(s) = f(s).$$

$$\text{When} \quad s = 0, v = u, \text{ so that } C = \frac{u^2}{2} - F(0).$$

$$\therefore \quad v^2 = u^2 + 2F(s) - 2F(0)$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{u^2 + 2F(s) - 2F(0)}$$

$$\therefore \quad \int dt = \int \frac{ds}{\sqrt{u^2 + 2F(s) - 2F(0)}}$$

$$\therefore \quad t = G(s) + C_1, \text{ where } \frac{d}{ds} G(s) = \frac{1}{\sqrt{u^2 + 2F(s) - 2F(0)}}.$$

$$\text{When} \quad t = 0, s = 0, \text{ so that } t = G(s) - G(0).$$

VI. Given  $a = f(v)$ , to express  $s$  and  $t$  in terms of  $v$ .

$$v \frac{dv}{ds} = f(v)$$

$$\therefore \quad \int ds = \int \frac{v dv}{f(v)},$$

$$\therefore \quad s = F(v) + C, \text{ where } \frac{d}{dv} F(v) = \frac{v}{f(v)}.$$

$$\text{When} \quad s = 0, v = u, \text{ so that } s = F(v) - F(u).$$

Suppose this relationship be expressed in the form

$$v = G(s)$$

$$\text{then} \quad \int \frac{ds}{G(s)} = \int dt$$

$$\therefore \quad t = H(s) - H(0), \text{ where } \frac{d}{ds} H(s) = \frac{1}{G(s)},$$

$$\therefore \quad t = H[F(v) - F(u)] - H(0).$$

NOTE I. When *one* relation is given between *any* two of the quantities  $t, s, v, a$ , the relation between *any* other two can be obtained. For example suppose the relation between  $v$  and  $s$  be given, then referring to case IV, the relation between  $a$  and  $v$  is obtained by eliminating  $s$  from the two equations  $v = f(s)$  and  $a = f'(s)$   $f(s)$ , and the relation between  $a$  and  $t$ , by eliminating  $s$  from  $a = f'(s)$   $f(s)$  and  $t = F(s) - F(0)$ .

Examples on the above work will be found after art. 8.

#### 4. Relative Velocity and Acceleration.

O is a fixed point on a line and P and Q are points movable on that line. Suppose that after a time  $t$ ,  $OP = x$  and  $PQ = y$ . Then the velocity and acceleration of Q relative to P are  $\frac{dy}{dt}$  and  $\frac{d^2y}{dt^2}$  and thus the velocity and acceleration of Q are  $\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$  and  $\left(\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}\right)$  respectively. If Q were nearer to O than P,  $y, \frac{dy}{dt}$  and  $\frac{d^2y}{dt^2}$  would be negative. If  $OQ = z$ , the

velocity and acceleration of  $Q$  relative to  $P$  would be  $\left(\frac{dz}{dt} - \frac{dx}{dt}\right)$  and  $\left(\frac{d^2z}{dt^2} - \frac{d^2x}{dt^2}\right)$  respectively. The relative velocity and acceleration of  $P$  with respect to  $Q$  have these magnitudes but the signs are reversed.

**5. Graphical Methods.**—When the law connecting two of the quantities  $t$ ,  $s$ ,  $v$  and  $a$  is not known and the relationship is determined only by a number of corresponding results tabulated as the result of experiment, the problems dealt with in Article 3 can be solved graphically.

I.

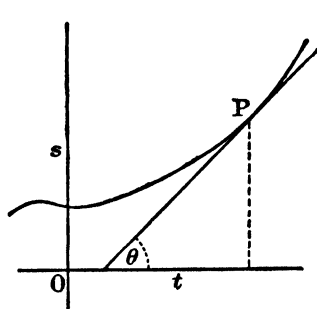


Fig. 1.

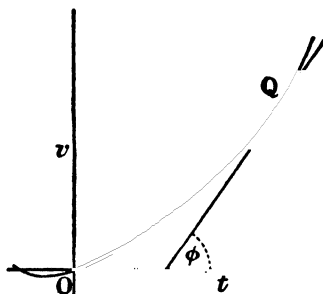


Fig 2.

Fig. 1 is the graph connecting  $s$  and  $t$ . The velocity at any time determined by a point  $P$  on the curve is the numerical value of  $\tan \theta$  as

$$\frac{ds}{dt} = \tan \theta.$$

The values of  $v$  for several values of  $t$  having been thus determined, the graph connecting  $v$  and  $t$  can be drawn (Fig. 2). Then the acceleration at any time determined by a point  $Q$  on the curve is given by  $\tan \phi$ .

II.

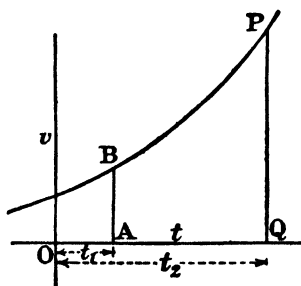


Fig. 3.

If  $v = f(t)$ ,  $\int ds = \int f(t) dt$  and the distance travelled between the instants when  $t = t_1$  and  $t = t_2$  is therefore  $\int_{t_1}^{t_2} f(t) dt$ . The value of this is given by the area  $AQP B$  which can either be obtained by Simpson's Rule, by counting squares or by the use of the planimeter.

III. If  $a = f(t)$  and when  $t = t_1$ ,  $v = v_1$ , then  $\int_{v_1}^v dv = \int_{t_1}^t f(t) dt$ ,

i.e., 
$$v = v_1 + \int_{t_1}^t f(t) dt.$$

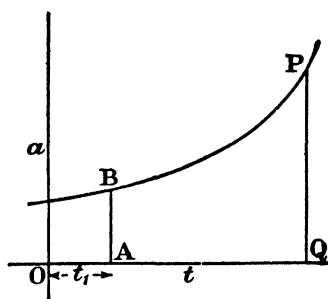


Fig. 4.

For any given value of  $t$ , the value of  $\int_{t_1}^t f(t) dt$  is given by the area AQP B. By taking P in different positions on the curve a set of corresponding values of  $v$  and  $t$  can be found. A graph can then be drawn and the distance found as in case II.

IV. If  $v = f(s)$ , then  $a = v \frac{dv}{ds} = f(s)f'(s)$ .

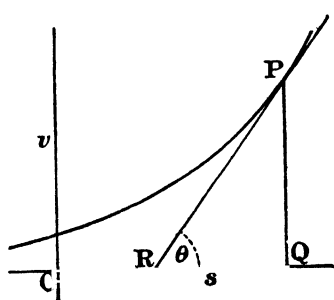


Fig. 5.

For the value of  $s$  determined by the point P on the curve, PQ gives the value of  $f(s)$  and  $\tan \theta$ , the value of  $f'(s)$ , therefore the value of  $a$  is determined by

$$PQ \tan \theta \text{ or } \frac{PQ^2}{RQ}.$$

If when  $t = t_1$ ,  $s = s_1$ ,

$$\text{then } t - t_1 = \int_{s_1}^s \frac{ds}{f(s)}.$$

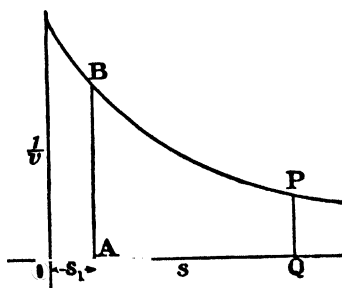


Fig. 6.

The value of  $t$  for any given value of  $s$  can be obtained by drawing the graph connecting  $s$  and  $\frac{1}{v}$  (i.e.  $\frac{1}{f(s)}$ ) for if P be the point on the curve (Fig. 6) corresponding to the value of  $s$  given, then the corresponding time is determined by the area AQP B plus  $t_1$ .

V. If  $a = f(s)$  and when  $s = s_1$ ,  $v = v_1$ , then

$$\int_{v_1}^v v dv = \int_{s_1}^s f(s) ds,$$



i.e., 
$$v^2 - v_1^2 = 2 \int_{s_1}^s f(s) ds.$$

The graph connecting  $a$  and  $s$  being drawn, the value of  $\int_{s_1}^s f(s) ds$  for any value of  $s$  can be determined by finding the area bounded by the  $s$  axis, the curve and the two ordinates.  $v^2$ , and therefore  $v$  is thus determined for any value of  $s$  and hence  $t$  can be found as in case IV.

VI. If  $a = f(v)$  and when  $s = s_1$ ,  $v = v_1$ ,

then 
$$\int_{s_1}^s ds = \int_{v_1}^v \frac{v dv}{f(v)},$$

i.e., 
$$s - s_1 = \int_{v_1}^v \frac{v dv}{f(v)}.$$

The graph connecting  $v$  and  $\frac{v}{a} \left( \text{i.e. } \frac{v}{f(v)} \right)$  being drawn, the value of  $\int_{v_1}^v \frac{v dv}{f(v)}$  for any value of  $v$  can be obtained and thus any number of

sets of corresponding values of  $v$  and  $s$  can be tabulated, and hence  $t$  can be found as in case IV.

**6. Motion with Constant Acceleration.**—Let  $s$  be the distance of the moving point and  $v$  its velocity after time  $t$  from a fixed point in the line of motion. Also let the velocity at the fixed point be  $u$  and the constant acceleration,  $f$ .

Then 
$$\frac{d^2s}{dt^2} = f.$$

Integrating with respect to  $t$ ,  $\frac{ds}{dt} = ft + C$  (a constant).

When  $t = 0$ ,  $\frac{ds}{dt} = u$ , so that  $u = C$

$\therefore \frac{ds}{dt} = u + ft \quad \text{i.e.} \quad v = u + ft.$

Integrating with respect to  $t$ ,  $s = ut + \frac{1}{2}ft^2 + C_1$  (a constant).

When  $t = 0$ ,  $s = 0$ , so that  $C_1 = 0$

$\therefore s = ut + \frac{1}{2}ft^2.$

Also 
$$v \frac{dv}{ds} = f.$$

Integrating with respect to  $s$ ,  $\frac{v^2}{2} = fs + C_1$  (a constant).

When  $s = 0$ ,  $v = u$ , so that  $\frac{u^2}{2} = C_1$ ,

$$\therefore v^2 = u^2 + 2fs.$$

**7. Simple Harmonic Motion.**—When a point moves in a straight line such that its acceleration is proportional to its distance from a fixed point in the straight line and is always directed towards the fixed point, its motion is said to be Simple Harmonic.

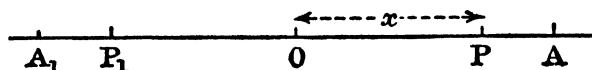


Fig. 7

O is the fixed point on the line  $A_1OA$  and P is the position of the point (moving in the  $+ve$  direction, i.e. from O to A) after time  $t$  from O. Let  $OP = x$  and the velocity at P,  $v$ .

Then 
$$v \frac{dv}{dx} = -\mu x.$$

Integrating with respect to  $x$ , we have

$$\frac{v^2}{2} = -\mu \frac{x^2}{2} + C \text{ (a constant).}$$

When  $v = 0$  let  $x = a$ , so that  $0 = -\mu \frac{a^2}{2} + C$ ,

$$\therefore v^2 = \mu(a^2 - x^2), \quad \dots \dots \dots (1)$$

$$\therefore v = \sqrt{\mu} \cdot \sqrt{a^2 - x^2},$$

as P is moving in the  $+ve$  direction.

As 
$$v = \frac{dx}{dt}, \quad \int \sqrt{\mu} \cdot dt = \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\therefore \sqrt{\mu} \cdot t = \sin^{-1} \frac{x}{a} + C_1 \text{ (a constant).}$$

When  $t = 0$ ,  $x = 0$  so that  $C_1 = 0$

$$\therefore x = a \sin \sqrt{\mu} \cdot t \quad \dots \dots \dots (2)$$

Let  $OA = a$ ,  $OA_1 = -a$  and  $OP_1 = -x$ .

The velocity of the moving point when at O is  $\pm \sqrt{\mu} \cdot a$  and at P,  $\pm \sqrt{\mu} \cdot \sqrt{a^2 - x^2}$ ; thus after the point reaches A it returns towards O and at P its velocity is the same as before but in the opposite direction. At  $P_1$  its velocity is  $\pm \sqrt{\mu} \sqrt{a^2 - x^2}$  and at  $A_1$  it is again zero. Thus the motion of the moving point is an oscillation between the points A and  $A_1$ .

$$\begin{aligned}
 \text{As } x &= a \sin \sqrt{\mu} \cdot t \\
 &= a \sin (\sqrt{\mu} \cdot t + 2\pi) \\
 &= a \sin \sqrt{\mu} \left[ t + \frac{2\pi}{\sqrt{\mu}} \right]
 \end{aligned}$$

the point has the same position, velocity and same direction after time  $t + \frac{2\pi}{\sqrt{\mu}}$  as it had after time  $t$ ; therefore the time

of a complete oscillation, i.e., the **Periodic Time** is  $\frac{2\pi}{\sqrt{\mu}}$ . The

distance OA or OA<sub>1</sub> is called the **Amplitude** and the number of complete oscillations per second is called the **Frequency**

which is therefore  $\frac{\sqrt{\mu}}{2\pi}$ .

**NOTE 2.** It can be easily verified that  $x = a \sin (\sqrt{\mu} t + g)$  is the solution of  $\frac{d^2x}{dt^2} = -\mu x$ ,  $a$  and  $g$  being arbitrary constants.  $x$  must lie between  $+a$  and  $-a$  so that  $a$  is the amplitude but unless  $g = 0$ , the time is not measured from the instant the moving point is at O. It should be noted that if the moving point be projected from O with different velocities, the amplitudes will be different but the periodic time will be the same.

**8. Compounding of Simple Harmonic Motions of the Same Period, the Accelerations being Proportional to the Displacements from the Same Fixed Point and the Motion being along the Same Straight Line.**

As the period is dependent only on the constant  $\mu$ ,  $v \frac{dv}{dx} = -\mu x$  is the equation of motion of all the separate simple harmonic motions.

Suppose the displacements of two of the simple harmonic motions to be  $a_1 \sin (\sqrt{\mu} t + g_1)$  and  $a_2 \sin (\sqrt{\mu} t + g_2)$  respectively, where  $a_1$ ,  $a_2$ ,  $g_1$  and  $g_2$  are constants determined by initial conditions. Then the resultant displacement is given by

$$\begin{aligned}
 x &= a_1 \sin (\sqrt{\mu} t + g_1) + a_2 \sin (\sqrt{\mu} t + g_2) \\
 &= \sin \sqrt{\mu} t \cdot (a_1 \cos g_1 + a_2 \cos g_2) + \cos \sqrt{\mu} t \cdot (a_1 \sin g_1 + a_2 \sin g_2)
 \end{aligned}$$

$$\text{Let } a_1 \cos g_1 + a_2 \cos g_2 = a \cos g$$

$$\text{and } a_1 \sin g_1 + a_2 \sin g_2 = a \sin g$$

$$\text{so that } a = (a_1^2 + a_2^2 + 2a_1a_2 \cos g_1 - g_2)^{\frac{1}{2}}$$

$$\text{and } g = \tan^{-1} \frac{a_1 \sin g_1 + a_2 \sin g_2}{a_1 \cos g_1 + a_2 \cos g_2}.$$

$$\text{Then } x = a \sin (\sqrt{\mu} t + g).$$

Thus the combined motion is simple harmonic and the amplitude is  $(a_1^2 + a_2^2 + 2a_1a_2 \cos g_1 - g_2)^{\frac{1}{2}}$ .

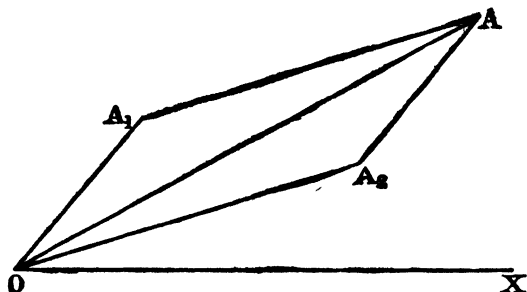


Fig. 8.

OX is a fixed line and the lines  $OA_1$  ( $OA_1 = a_1$  and  $\widehat{A_1OX} = g_1$ ) and  $OA_2$  ( $OA_2 = a_2$  and  $\widehat{A_2OX} = g_2$ ) represent (by determining the constants *amplitude* and *phase*) the simple harmonic motions respectively. The diagonal OA of the completed parallelogram has length  $a$  and  $\widehat{AOX} = g$  and thus OA represents the resultant simple harmonic motion.

By the substitutions

$$a_1 \cos g_1 + a_2 \cos g_2 + \dots + a_n \cos g_n = a \cos g$$

$$\text{and } a_1 \sin g_1 + a_2 \sin g_2 + \dots + a_n \sin g_n = a \sin g$$

$n$  simple harmonic motions can be compounded into one simple harmonic motion.

**Ex. I.** A particle moves along a straight line so that after  $t$  sec. its distance  $s$  from a fixed point, O, on the line is given by  $s = (t-1)^2(t-2)$ . Find (1) the velocity and acceleration on each occasion when it passes through O; (2) its distances from O when its velocity is zero; (3) its distance from O when its acceleration is zero; (4) its acceleration when its velocity is 8 ft. per sec. per sec. Describe the motion.

$$s = t^3 - 4t^2 + 5t - 2$$

$$\therefore \frac{ds}{dt} = 3t^2 - 8t + 5$$

$$\text{and } \frac{d^2s}{dt^2} = 6t - 8.$$

(1) The particle passes through O when  $t = 1$  or 2.

After 1 sec. its vel. is zero and its accel. =  $-2$  ft./sec.<sup>2</sup>

" 2 sec. " " " " " " " = 4 ft./sec.<sup>2</sup>

(2) Its vel. is zero when  $3t^2 - 8t + 5 = 0$ , i.e. when  $t = 1$  or  $1\frac{2}{3}$ .

Its distance from O at the end of 1 sec. is zero

" " " " " " " " "  $1\frac{2}{3}$  sec. "  $-\frac{4}{27}$  ft.

(3) Its accel. is zero when  $t = 1\frac{1}{3}$

Its distance from O at the end of  $1\frac{1}{3}$  sec. is  $-\frac{2}{27}$  ft.

(4) Its vel. is 8 ft./sec. when  $3t^2 - 8t + 5 = 8$ , i.e., when  $t = 3$  or  $-\frac{1}{3}$ .

Its accel. after 3 sec. = 10 ft./sec.<sup>2</sup>

„ „  $\frac{1}{3}$  sec. before zero time = - 10 ft./sec.<sup>2</sup>

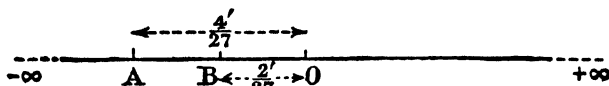


Fig. 9.

From  $t = -\infty$  to  $t = 1$ , the particle moves from  $-\infty$  to O with decreasing speed and is instantaneously at rest at O. From  $t = 1$  to  $t = 1\frac{1}{3}$  it moves from O to B with increasing speed and from  $t = 1\frac{1}{3}$  to  $t = 1\frac{2}{3}$  it moves from B to A with decreasing speed. From  $t = 1\frac{2}{3}$  to  $t = \infty$  it moves with increasing speed from A to  $+\infty$ .

**Ex. II.** A particle is projected with velocity  $u$  ft. per sec. from a fixed point O on a straight line, and moves along the line so that its acceleration ( $a$ ) is given by  $k\sqrt{t}$  where  $t$  is the time from the instant it leaves O and  $k$  is a constant. Express the distance from O in terms of the acceleration. If the numerical value of  $k$  be 2 find  $u$  in order that when the particle has moved 1 ft. its acceleration shall be 1 ft./sec.<sup>2</sup>. Under these latter conditions find the velocity after 4 sec.

Let  $s \equiv$  dist. of the particle from O after time  $t$ .

Then 
$$a = \frac{d^2s}{dt^2} = k\sqrt{t}$$

$\therefore \frac{ds}{dt} = \frac{2k}{3} \cdot t^{\frac{3}{2}} + u$  for  $\frac{ds}{dt} = u$  when  $t = 0$

$\therefore s = \frac{4k}{15} t^{\frac{5}{2}} + ut$  for  $s = 0$  when  $t = 0$ ,

but  $t^{\frac{1}{2}} = \frac{a}{k}$  so that  $s = \frac{4a^5}{15k^4} + \frac{ua^2}{k^2}$ .

Substituting  $k = 2, s = 1, a = 1$ ,

$$1 = \frac{4}{15 \times 16} + \frac{u}{4}$$

i.e.  $u = 3\frac{1}{3}$ .

Velocity after 4 sec.  $= \frac{4}{3} \times 8 + 3\frac{1}{3}$  ft. per sec.  
 $= 14\frac{2}{3}$  ft. per sec.

**Ex. III.** A particle initially at rest moves from a fixed point in a straight line so that at the end of time  $t$  sec. its acceleration is  $3 \sin t + \frac{1}{(t+1)^2}$  ft./sec.<sup>2</sup>. What is its distance from the fixed point at the end of 4 sec.?

Let  $s$  ft. be the distance from the fixed point after  $t$  sec.

Then 
$$\frac{d^2s}{dt^2} = 3 \sin t + \frac{1}{(t+1)^2}$$

$\therefore \frac{ds}{dt} = -3 \cos t - \frac{1}{(t+1)} + 4$  as  $\frac{ds}{dt} = 0$  when  $t = 0$ .

$$\therefore s = -3 \sin t - \log_e (t + 1) + 4t \quad \text{as } s = 0$$

when  $t = 0$ .

$$\text{When } t = 4, s = -3 \sin (4 \times 57^\circ 18') - \log_e 5 + 16$$

$$= 3 \sin 49^\circ 12' - 1.609 + 16$$

$$= 16.66$$

**Ex. IV.** A body moving in a straight line has described the following distances at the given times.

|                      |   |       |    |        |     |        |      |
|----------------------|---|-------|----|--------|-----|--------|------|
| Time in seconds ...  | 0 | 5     | 10 | 15     | 20  | 25     | 30   |
| Distance in feet ... | 0 | 13.75 | 60 | 176.25 | 400 | 768.75 | 1320 |

Deduce approximately the velocity-time and acceleration-time curves for the same period.

| $t$ | $s$    | $\delta s$ | $\frac{\delta s}{\delta t}$ i.e.<br>approx. value<br>of $\frac{ds}{dt}$ | $\delta \left( \frac{ds}{dt} \right)$ | $\frac{\delta \left( \frac{ds}{dt} \right)}{\delta t}$ i.e.<br>approx. value<br>of $\frac{d^2s}{dt^2}$ | $\delta \left( \frac{d^2s}{dt^2} \right)$ |
|-----|--------|------------|-------------------------------------------------------------------------|---------------------------------------|--------------------------------------------------------------------------------------------------------|-------------------------------------------|
| 0   | 0      |            |                                                                         |                                       |                                                                                                        |                                           |
| 5   | 13.75  | 13.75      | 2.75                                                                    | 6.5                                   | 1.3                                                                                                    |                                           |
| 10  | 60     | 46.25      | 9.25                                                                    | 14.0                                  | 2.8                                                                                                    | 1.5                                       |
| 15  | 176.25 | 116.25     | 23.25                                                                   | 21.5                                  | 4.3                                                                                                    | 1.5                                       |
| 20  | 400    | 223.75     | 44.75                                                                   | 29.0                                  | 5.8                                                                                                    | 1.5                                       |
| 25  | 768.75 | 368.75     | 73.75                                                                   | 36.5                                  | 7.3                                                                                                    | 1.5                                       |
| 30  | 1320   | 551.25     | 110.25                                                                  |                                       |                                                                                                        |                                           |

The velocity-time curve is obtained by plotting—

|                         |      |      |       |       |       |        |
|-------------------------|------|------|-------|-------|-------|--------|
| Time in sec. ...        | 2.5  | 7.5  | 12.5  | 17.5  | 22.5  | 27.5   |
| Velocity in ft/sec. ... | 2.75 | 9.25 | 23.25 | 44.75 | 73.75 | 110.25 |

The acceleration-time curve is obtained by plotting—

|                                         |     |     |     |     |     |
|-----------------------------------------|-----|-----|-----|-----|-----|
| Time in sec. ...                        | 5   | 10  | 15  | 20  | 25  |
| Acceleration in ft/sec <sup>2</sup> ... | 1.3 | 2.8 | 4.3 | 5.8 | 7.3 |

NOTE 3. The approx. values of  $\frac{ds}{dt}$  in the table are the slopes of the lines joining the points on the curve  $t = 0, t = 5; t = 5, t = 10$ , etc., and this gives an average value of  $\frac{ds}{dt}$  in the respective time periods, so that approx.

values of  $\frac{ds}{dt}$  are obtained for values of  $t, 2.5, 7.5, \dots, 27.5$ . More accurate results could have been obtained by drawing the distance-time graph and constructing the tangents to the curve at the points  $t = 0, t = 5 \dots t = 30$  and evaluating the slopes which will be the values of  $\frac{ds}{dt}$  corresponding to  $t = 0, t = 5 \dots t = 30$ .

The acceleration-time curve is derived from the velocity-time curve in the same way as the latter is derived from the distance-time curve.

NOTE 4. In this particular case it is seen that  $\frac{d^2s}{dt^2} = 0.3$  and thus the exact law between  $s$  and  $t$  is known. Integrating with respect to  $t$ , successively, three times then  $s = 0.05t^3 + c_1t^2 + c_2t + c_3$ , where  $c_1, c_2, c_3$  are constants which can be evaluated by substituting any three sets of corresponding values of  $s$  and  $t$ . Substituting  $\begin{vmatrix} t & 0 & 10 & 20 \\ s & 0 & 60 & 400 \end{vmatrix}$  then  $c_3 = 0, 10c_1 + c_2 = 1, 20c_1 + c_2 = 0$  and hence  $s = 0.05t^3 - 0.1t^2 + 2t, \frac{ds}{dt} = 0.15t^2 - 0.2t + 2$  and  $\frac{d^2s}{dt^2} = 0.3t - 0.2$ .

EX. V. A straight rod 20' long has its ends A and B in straight grooves in the same plane and at right angles to one other. Initially the angle between the rod and the groove along which B can move is  $60^\circ$  and the end B is made to move from rest towards O with an acceleration equal to  $4 \text{ OB}$  where O is the point of intersection of the grooves. Find the acceleration of the end A when B reaches O, and find the position of A where its velocity is a maximum.

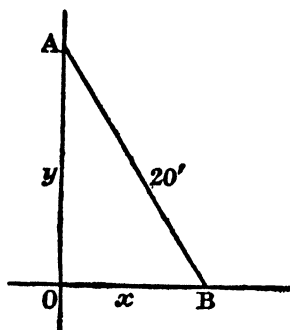


Fig. 10.

At the end of time  $t$ , let  $OB = x$ ,

$$OA = y \text{ and } \frac{dx}{dt} = v.$$

$$\text{Then } y = (400 - x^2)^{\frac{1}{2}} \dots \dots (1)$$

$$\text{and } v \frac{dv}{dx} = -4x \dots \dots (2)$$

$$\text{From (2), } v^2 = -4x^2 + C.$$

$$\text{When } x = 10, v = 0, \text{ so that } C = 400.$$

$$\therefore v^2 = 4(100 - x^2).$$

Since  $x$  decreases as  $t$  increases,  $v$  is negative.

$$\therefore \frac{dx}{dt} = -2(100 - x^2)^{\frac{1}{2}} \dots \dots (3)$$

$$\text{From (1), } \frac{dy}{dt} = \frac{-x}{(400 - x^2)^{\frac{1}{2}}} \cdot \frac{dx}{dt}$$

$$= \frac{2x(100 - x^2)^{\frac{1}{2}}}{(400 - x^2)^{\frac{1}{2}}} \quad \text{from (3)}$$

$$\text{i.e.} \quad \left(\frac{dy}{dt}\right)^2 = \frac{4x^2(100 - x^2)}{(400 - x^2)}$$

$$\therefore 2 \left(\frac{dy}{dt}\right) \frac{d^2y}{dt^2} = \frac{8[(400 - x^2)(100x - 2x^3) + x^3(100 - x^2)]}{(400 - x^2)^2} \cdot \frac{dx}{dt}$$

$$\therefore \frac{d^2y}{dt^2} = \frac{-4[(400 - x^2)(100 - 2x^2) + x^2(100 - x^2)]}{(400 - x^2)^{\frac{3}{2}}}$$

When B reaches O, the acceleration of A is  $\frac{-4 \times 40000}{8000} \text{ ft/sec}^2$

i.e.  $20 \text{ ft/sec}^2$  towards O.

The velocity of A is a maximum when  $\frac{d^2y}{dt^2} = 0$ ,

$$\text{i.e. when} \quad x^4 - 800x^2 + 40000 = 0$$

$$\text{i.e. ,,} \quad (400 - y^2)^2 - 800(400 - y^2) + 40000 = 0$$

$$\text{i.e. ,,} \quad y^4 - 120000 = 0.$$

Thus A has maximum velocity at a distance of  $10.12^{\frac{1}{2}}$  ft. from O.

**Ex. VI.**—A particle is projected with velocity  $V$  directly away from a fixed point at a distance  $b$  from the point of projection. If the acceleration be  $\mu \times$  (distance from the fixed point) and always directed towards the fixed point, find the amplitude of the S.H.M.

Let  $x \equiv$  distance of the particle from the fixed point at the end of time  $t$ , and  $v \equiv$  velocity of the particle after time  $t$ .

$$\text{Then} \quad v \frac{dv}{dx} = -\mu x$$

$$\therefore \quad v^2 = -\mu x^2 + C.$$

$$\text{When } x = b, v = V, \text{ so that } V^2 = -\mu b^2 + C.$$

$$\therefore \quad v^2 = V^2 + \mu(b^2 - x^2).$$

The value of  $x$  when  $v$  is zero determines the amplitude.

$$\text{When } v = 0, \quad x^2 = b^2 + \frac{V^2}{\mu}.$$

**Ex. VII.** On a given day the depth at high water over a harbour bar is 32 feet and at low water  $6\frac{1}{4}$  hours later it is 20 feet.

If high water is due at 7.10 p.m., what is the latest time that day at which a ship drawing 28 feet can cross the bar, assuming the rise and fall of the tide to be simple harmonic?

Let  $t \equiv$  time in hr. for the level to change from 28' to 26'.

Let  $\mu x \equiv$  acceleration of water level in ft. per hr.,  $x$  being the distance in ft. from mean level (26').



|     |              |                                                                                           |
|-----|--------------|-------------------------------------------------------------------------------------------|
| 32  | Then         | $\frac{2\pi}{\sqrt{\mu}} = \frac{25}{2}$                                                  |
| 28' | $\therefore$ | $2 = 6 \sin \frac{4\pi}{25} t$                                                            |
| 26' | $\therefore$ | $\sin \frac{4\pi}{25} t = .3333$                                                          |
|     | $\therefore$ | $\frac{4\pi}{25} t = 19^\circ 28' = 19\frac{7}{8} \times \frac{\pi}{180} \text{ radians}$ |
| 20' | $\therefore$ | $t = \frac{292 \times 25}{15 \times 180 \times 4} \text{ hr.}$                            |
|     |              | $= 40 \text{ min. } 33 \text{ sec. (to nearest sec.)}$                                    |

The tide falls to mean level at 10 hr. 17 min. 30 sec. p.m.

$\therefore$  latest time for ship to cross bar is 9 hr. 36 min. 57 sec. p.m.

**Ex. VIII.** A motor-car passes a mile stone and its subsequent speed ( $v$  miles per hour) is given by  $v = \frac{40s + 24}{s + 1}$ , where  $s$  miles is the distance travelled from the mile stone. How long will it take to travel 10 miles from the mile stone and what will its acceleration be then?

$t$  hours after leaving the mile stone, we have

$$\frac{ds}{dt} = \frac{40s + 24}{s + 1},$$

$$\therefore t = \int \frac{s + 1}{40s + 24} ds$$

$$= \frac{1}{40} \int \left[ 1 + \frac{16}{40s + 24} \right] ds;$$

$$\therefore 40t = s + \frac{2}{5} \log_e(5s + 3) + C \text{ (const.)}.$$

When  $t = 0, s = 0$  so that  $C = -\frac{2}{5} \log 3$

$$\therefore 40t = s + \frac{2}{5} \log_e\left(\frac{5}{3}s + 1\right).$$

Time taken to travel 10 miles  $= \frac{1}{40} \left[ 10 + \frac{2}{5} \log_e 17.67 \right] \text{ hr.}$

$$= \frac{1}{40} \left[ 10 + \frac{2 \times 1.2472}{5 \times .4343} \right] \text{ hr.}$$

$$= 16 \text{ min. } 43 \text{ sec.}$$

$$v \frac{dv}{ds} = \frac{8(5s + 3)}{s + 1} \cdot \frac{16}{(s + 1)^2} = \frac{128(5s + 3)}{(s + 1)^3}$$

$$\therefore \text{ accel. after travelling 10 miles } = \frac{128 \times 53}{11^3} \text{ ml./hr.}^2$$

$$= 5.097 \text{ ml./hr.}^2$$

**Ex. IX.** The retardation due to frictional resistance to the motion of a train moving on the level with velocity  $v$  ft. per sec., is given approximately by  $a + bv^2$  ft. per sec. per sec. Deduce from this a formula for the whole distance gone by a train after steam has been shut off, the initial speed being  $V$ . Evaluate the result, taking  $a = 0.085$ ,  $b = 0.00027$  and  $V = 44$ .

Let  $s$  ft. be the distance the train travels from the instant its velocity is  $V$  ft./sec. to that when it is  $v$  ft./sec.

Then 
$$v \frac{dv}{ds} = -(a + bv^2)$$

$\therefore \int ds = - \int \frac{v dv}{a + bv^2}$

$\therefore s = -\frac{1}{2b} \log_e (a + bv^2) + C.$

When  $s = 0$ ,  $v = V$ , so that  $C = \frac{1}{2b} \log_e (a + bV^2)$

$\therefore s = \frac{1}{2b} \log_e \frac{a + bV^2}{a + bv^2}.$

The whole distance travelled by the train is  $\frac{1}{2b} \log_e \left( 1 + \frac{b}{a} V^2 \right)$  ft.

When  $\begin{cases} a = 0.085 \\ b = 0.00027, \\ V = 44 \end{cases} \quad \frac{b}{a} V^2 = 6.150$

$\therefore \text{distance travelled} = \frac{.8543}{.00054 \times .4343} \text{ ft.}$   
 $= 3643 \text{ ft.}$

### Problems I

1. A particle moves along a straight line with acceleration  $3t + 2$  ft./sec.<sup>2</sup> where  $t$  is the time in sec. after leaving a fixed point. At the end of 1 sec. its distance from the fixed point is 10 ft. and after 2 sec., 24 ft. What is the velocity and the distance of the particle from the fixed point at the end of 3 sec.?

2. A man walks so that after  $t$  hours his speed is  $\frac{20}{5+t}$  miles per hour. How far has he walked by the time his speed is reduced to 3 miles per hour and how long does he take to walk 10 miles?

3. A particle moves in a straight line from a fixed point O and its distance ( $s$  ft.) from O after  $t$  sec. is such that  $s = e^{2t} \sin 3t$ . Find its initial velocity and acceleration and its distance from O when its velocity is for the first time zero and for the second time zero.

4. A particle starts with given velocity  $V$  and moves under a retardation equal to  $k$  times the space described. Show that the distance traversed before it comes to rest is  $V/\sqrt{k}$ .

5. A particle starts from rest at a distance  $a$  from a fixed point O and moves towards O with an acceleration  $k \left( s + \frac{a^4}{s^3} \right)$  where  $s$  is its distance from O. Show that it will reach O in time  $\frac{\pi}{4\sqrt{k}}$ . Also find the time it takes to reach a point at a distance  $\frac{a}{\sqrt{2}}$  from O.

6. A man starts on a long walk at 4 ml. per hr. and his speed at any instant is inversely proportional to  $x + 10$ , where  $x$  is the number of miles he has walked. Find (1) the time he takes to walk 20 ml., (2) the distance he goes in 4 hr. and his speed then.

7. A steamboat is moving at velocity  $V$  when steam is shut off. Given that the retardation when the velocity is  $v$  ft. per sec. is  $\cdot 00068 v^3$ , find the distance the steamboat will have travelled by the time its velocity is  $\frac{1}{2}V$ .

8. An arc lamp is 18 feet from the ground. A man  $5' 7\frac{1}{2}"$  tall walks 4 miles per hour along a line whose nearest distance from the base of the lamp post is 6 feet. Express the velocity of the end of his shadow  $t$  seconds from the time he is nearest the lamp.

9. By proper choice of units the curve on a time base representing the acceleration of an electric train is a quadrant of a circle, whose centre is the origin. The initial acceleration is 2.5 ft. per sec. per sec., and the acceleration falls to zero in 20 seconds. Calculate the velocity acquired and the distance described in that time.

10. The speed of a motor car travelling on an undulating road is read by an indicator at the end of each minute and the speeds are :—

|                       |     |     |    |    |    |    |      |    |    |      |   |    |    |    |    |    |    |
|-----------------------|-----|-----|----|----|----|----|------|----|----|------|---|----|----|----|----|----|----|
| Time (mins.)          | ... | 0   | 1  | 2  | 3  | 4  | 5    | 6  | 7  | 8    | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| Speed (mils. per hr.) | 0   | 7.5 | 11 | 15 | 20 | 22 | 17.5 | 21 | 10 | 12.5 | 5 | 18 | 20 | 20 | 18 | 14 |    |

Determine the total distance travelled and the average speed.

11. A train moves from rest and the accelerations at different times are given as follows :—

|                                  |     |     |   |     |   |     |    |   |
|----------------------------------|-----|-----|---|-----|---|-----|----|---|
| Time (sec.)                      | ... | ... | 0 | 2   | 3 | 4   | 5  | 6 |
| Accel : (ft./sec. <sup>2</sup> ) | ... | 1   | 7 | 8.2 | 9 | 9.6 | 10 |   |

Find graphically the velocity at the end of 6 seconds.

12. A particle moves along a straight line so that  $\frac{d^3s}{dt^3}$  is always 6 ft./sec.<sup>3</sup>. At the end of 1 sec. the distance from the origin is 2 ft., the velocity is 2 ft./sec. and the acceleration 2 ft./sec.<sup>2</sup>. Find the distance, velocity and acceleration at the end of 2 sec. Also prove

that the velocity ( $v$  ft./sec.) and the acceleration ( $a$  ft./sec.<sup>2</sup>) satisfy the relation  $12v = a^2 + 20$  for all values of  $t$ .

13. A body moving in a straight line describes the following distances in the given times :—

|                     |   |      |      |     |       |     |
|---------------------|---|------|------|-----|-------|-----|
| Time in sec. ... .. | 0 | 10   | 17   | 23  | 28    | 35  |
| Distance in ft. ... | 0 | 41.8 | 98.5 | 162 | 221.5 | 316 |

Construct the velocity-time and acceleration-time curves for the same period. Find the velocity and the acceleration at the end of 20 sec.

14. A train travels from A to B, a distance of 8 miles with gradually increasing velocity. From the following table deduce the time taken to the nearest  $\frac{1}{10}$  min.

|                                   |    |    |    |      |    |    |      |       |       |
|-----------------------------------|----|----|----|------|----|----|------|-------|-------|
| Distance from A in miles ... ..   | 0  | 1  | 2  | 3    | 4  | 5  | 6    | 7     | 8     |
| Velocity in miles per hour ... .. | 20 | 35 | 40 | 42.5 | 44 | 45 | 45.7 | 46.25 | 46.67 |

15. A body moves from rest from a point O so that its acceleration is  $\frac{1}{(t+1)^2}$  where  $t$  is the time taken in seconds, from O. Find the distance moved in 9 seconds and its velocity then.

16. A ladder 30' long slides down from a vertical wall, remaining in a vertical plane. What is the velocity of the upper end when the lower end is at a distance of 5 feet from the base of the wall and has a velocity of 2 feet per second?

17. A particle moves in a straight line from a fixed point O with velocity  $V$  under a force which produces an acceleration  $\mu x$  where  $x$  is its distance from O. Find the time taken in moving a distance  $x$  and the distance moved in time  $t$ . Also find the time taken for the velocity to be increased to  $2V$ .

18. A particle oscillates on the axis of  $x$  in such a way that its abscissa after  $t$  seconds is  $ae^{-\mu t} \sin qt$ . Find the times when it is at maximum distance from the origin.

Prove that the time taken in going out from the origin to a maximum distance is less than the time taken in coming back in the ratio of  $\tan^{-1} \frac{q}{p}$  to  $\pi - \tan^{-1} \frac{q}{p}$ . If these times are respectively  $\frac{1}{15}$  sec. and  $\frac{1}{15}$  sec., prove that the successive maximum distances form a geometrical progression whose common ratio is nearly 0.82.

19. A conical shaped vessel, semi-vertical angle  $30^\circ$ , has its axis vertical and vertex downwards. Water is poured into it at the rate of 1 pint per second. Given that 1 pint = 34.66 c. in., find the velocity of the surface of the water when the depth is  $x$  inches.

20. Given that the acceleration of a point is  $a - bv^3$  and that when  $t = 0$ ,  $s = 0$  and  $v = 0$ , prove that :—

(i.) Acceleration expressed as a function of  $s$  is  $a e^{-2bs}$ .

(ii.) Velocity " " " "  $t$  "  $\sqrt{\frac{a}{b}} \tanh(\sqrt{ab} \cdot t)$

(iii) Distance      "      "      "      "       $t$        $\frac{1}{b} \log \cosh \left( \sqrt{ab} . t \right)$

21. A particle is moving with simple harmonic motion, and while making an excursion from one position of rest to another, its distances from the middle point of its path at three consecutive seconds are observed to be  $x_1, x_2, x_3$ ; prove that the time of a complete oscillation is

$$\frac{2\pi}{\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)}.$$

22. A particle moves in a straight line under an attractive force towards a fixed point O in the line, so that the acceleration produced is  $\omega^2$  (distance from O), where  $\omega$  is constant. If at time  $t = 0$  the particle is at a distance  $b$  from O and is moving towards O with speed  $v$ , find the time it takes to reach O.

If, when the particle is next at the farthest distance from O, it receives an impulse which starts it back towards O with velocity  $v$ , show that the whole time that has elapsed when it reaches O a second time is

$$\frac{I}{\omega} \left\{ a + \frac{\pi}{2} + \tan^{-1} (\sec a) \right\},$$

where  $\tan \alpha = \frac{b\omega}{\eta}$ .

23. A particle P of mass  $m$  moves in a straight line L so that its displacement  $x$  from a fixed point O of the line is  $x = a \sin(pt + \epsilon)$ , where  $a, p, \epsilon$  are constants. Prove that the force on the particle is an attraction towards O of magnitude  $mp^2x$ .

When the particle P is passing through O it is overtaken by a second particle of mass  $2m$  moving with speed  $2ap$  in the line L and in the same direction as P. The particles coalesce on impact and move thereafter under the same force of attraction towards O as previously acted on P. Prove that the amplitude of the motion of the composite particle is  $5a\sqrt{3}/3$  and find the period.

## CHAPTER II

### LINEAR MOTION II

**9. Newton's Law of Motion.**—I. Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it be compelled by impressed force to change that state.

II. The rate of change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts.

III. To every action there is an equal and opposite re-action.

**10. Measurement of Forces.**—Law II affords a means by which force can be measured, for if  $F$  is the force acting on mass  $m$  which is moving with velocity  $v$ , then

$$F = k \frac{d}{dt} (mv) \text{ where } k \text{ is a constant}$$

i.e. 
$$F = k \left[ m \frac{dv}{dt} + v \frac{dm}{dt} \right].$$

If  $m$  is constant,  $\frac{dm}{dt} = 0$  and thus  $F = kma$ , where  $a$  is

the acceleration of the mass. Suppose the unit of force be so chosen that unit force acting on unit mass produces unit acceleration, then  $k = 1$  and thus  $F = ma$ .

The unit of force, for the F.P.S. system is called a **Poundal**, and that for the C.G.S. system is called a **Dyne**. The acceleration ( $g$ ) of a freely falling body at the Earth's surface for the latitude of London is approximately 32.2 feet per second per second or 981 centimetres per second per second and thus 1 lb. weight = 32.2 poundals and 1 gram weight = 981 dynes.

Law I is the particular case of Law II when  $F = 0$ .

An impulsive force is a very great force acting for a very small time and cannot in general be measured by the above method for the time it is acting, and therefore the acceleration of the mass on which it is acting cannot be determined to a sufficient degree of accuracy. The effect of such a force is measured by the **Impulse** of the force during the time of its action. Suppose that a force  $F$  acts on a mass during a time

$T$ , then the impulse of the force is defined by  $\int_0^T F dt$ .

For a constant mass  $m$ ,  $F = m \frac{dv}{dt}$ , therefore the impulse

$$= \int_0^T m \frac{dv}{dt} dt = m \int_U^V dv = mV - mU \text{ where } U \text{ is the initial}$$

velocity of the mass and  $V$  its velocity after time  $T$ . Thus the impulse of the force equals the momentum generated in the direction of the force in time  $T$ .

**11. Conservation of Linear Momentum.**—Law III states that if two masses act on each other in any way, either by impact, or by connection with strings or rods, or by mutual attraction or repulsion, the force exerted by one mass on the other has the same magnitude as, but is opposite in direction to, that which the other body exerts on the one.

Suppose that a system of bodies be mutually acting one on the other, but free from extraneous forces. Let the forces acting in the positive direction be  $F_1, F_2, \dots$  then the forces acting in the negative direction will be  $F_1, F_2, \dots$ . In time  $T$  the gain in momentum in the positive direction is

$$\int_0^T F_1 dt + \int_0^T F_2 dt + \dots$$

and the gain in the negative direction is also

$$\int_0^T F_1 dt + \int_0^T F_2 dt + \dots$$

Hence it follows that, **if no extraneous forces act on a system of bodies, the total momentum of the system in any assigned direction remains constant.**

Let  $x$  be the distance of a particle of mass  $m$  and  $\bar{x}$  the distance of the centre of inertia of the system from any fixed point on the line along which the particles of the system move, then

$$\Sigma(m) \cdot \bar{x} = \Sigma(mx)$$

$$\therefore \Sigma(m) \frac{d\bar{x}}{dt} = \Sigma\left(m \frac{dx}{dt}\right).$$

Now  $\Sigma\left(m \frac{dx}{dt}\right)$  is the total linear momentum and is there-

fore constant and therefore also  $\frac{d\bar{x}}{dt}$  is constant. Thus **if no**

**extraneous forces act on a system of particles moving along a straight line the centre of inertia is either at rest or moves with uniform velocity.**

**12. Work, Energy and Power.**—A Force is said to do work when it moves its point of application in the direction of the force. Suppose  $F$  to be a force acting in a constant direction on a body which moves in this direction from  $A$  to  $B$ . Suppose the distances of  $A$  and  $B$  from a fixed point on the line  $AB$  to be  $a$  and  $b$  respectively, then the work done by the force is  $\int_a^b F dx$ . If the force  $F$  be constant the work done will be

$F(b - a)$ . The F.P.S. unit of work is the **Foot-Poundal** which is the work done by a poundal in moving its point of application through one foot. The practical unit is the **Foot-Pound** ( $= 32.2$  foot-poundals) and the C.G.S. unit of work is called an **Erg** which is the work done by a dyne in moving its point of application through 1 centimetre.

The Energy of a body is its capacity for doing work.

The **Kinetic Energy** of a body is the energy it possesses by virtue of its motion, and is measured by the amount of work that the body can perform against the impressed forces before it is brought to rest.



Suppose a particle of mass  $m$  has a velocity  $U$  when at  $A$  and moves from  $A$  to  $B$  against a force  $f(x)$ , where  $x$  is the distance of the particle at any time from a fixed point  $O$  on the line  $AB$ . Let the velocity of the particle when at  $B$  be  $V$ . The work done by the body against the force is  $\int_a^b f(x) dx$ , where  $OA = a$  and  $OB = b$ . Now at any instant

$$mv \frac{dv}{dx} = -f(x).$$

$$\therefore \int_a^b f(x) dx = -m \int_U^V v dv = \frac{1}{2} m U^2 - \frac{1}{2} m V^2.$$

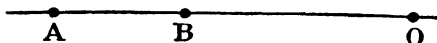
If the particle be brought to rest at  $B$ ,  $V = 0$  and thus the kinetic energy of the particle at  $A$ , i.e. when its velocity is  $U$ , is  $\frac{1}{2} m U^2$ .

Suppose the force to act in the direction  $OA$ , then  $mv \frac{dv}{dx} = f(x)$ .

The work done by the force in moving its point of application from  $A$  to  $B$  is  $\int_a^b f(x) dx = m \int_U^V v dv = \frac{1}{2} m V^2 - \frac{1}{2} m U^2$  which is the K.E. of the particle at  $B$  minus the K.E. at  $A$ . Thus the **work done by the force equals the kinetic energy produced**.



The **Potential Energy** of a body is the work it can do by virtue of its position in passing from this position to some standard position, or zero position. A compressed spring, a stretched elastic string and a body some distance from the earth's surface each possess potential energy.



Suppose a particle moves from rest at A towards a fixed point O under the action of a conservative force. Such a force is independent of the velocity and is entirely dependent on the position of the particle. (Friction is not a conservative force for its direction is dependent on the velocity as far as direction is concerned). Suppose, when at distance  $x$  from A the force is  $F(x)$ . Let  $AO = a$ ,  $AB = x$  and the potential energy of the particle at O, C. Then the potential energy of the particle at A is

$$\int_0^a F(x) dx + C$$

which equals  $\int_0^x F(x) dx + \int_x^a F(x) dx + C$ .

Now  $\int_x^a F(x) dx + C$  is the potential energy of the particle at B and  $\int_0^x F(x) dx$  is the kinetic energy at B. Thus the sum of the kinetic energy and potential energy of the particle at any point during the motion equals the potential energy at A where the kinetic energy is zero. In other words, when a particle is under the action of a conservative force, the sum of the potential and kinetic energies is a constant quantity.

**13. Impact, Newton's Experimental Law.**—By experiment, Newton discovered that, if two elastic bodies impinge directly, the velocity with which they separate after impact bears a constant ratio to the velocity with which they approached one another before impact. This constant ratio depends on the substances of which the bodies are composed and is independent of the masses of the bodies. It is generally denoted by  $e$  and is termed the **Coefficient of Restitution**.

If a body of mass  $m$  moving with velocity  $u$  overtakes a body of mass  $m'$  moving with velocity  $u'$ , then by the conservation of linear momentum, if  $v$  and  $v'$  denote the respective velocities measured in the same direction after impact,

$$mv + m'v' = mu + m'u' \quad \dots \quad \dots \quad (1)$$

and by Newton's experimental law regarding impact,

$$v' - v = e(u - u') \quad \dots \quad \dots \quad (2)$$

From these two equations,  $v$  and  $v'$  can be determined.

Unless the bodies are completely elastic  $e < 1$  so that  $u - u' > v' - v$

i.e.,

From (x),

∴

i.e.

and thus kinetic energy is lost by impact.

$$u + v > u' + v'.$$

$$m(u - v) = m'(v' - u')$$

$$m(u^2 - v^2) > m'(v'^2 - u'^2)$$

$$\frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 > \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2$$

**14. Hooke's Law.**—This law states that the tension of an elastic string or spring is proportional to the extension of the string beyond its natural length.

In the case of a light string hanging vertically or a heavy string resting on a smooth horizontal table, the tension will be the same throughout.

Let  $l$  be the natural length of a string and let  $T$  be its tension when it is stretched to be of length  $x$ ; then  $T = \frac{\lambda}{l}(x - l)$ ,

where  $\lambda$  is a constant called the **Modulus of Elasticity of the String**. The value of  $\lambda$  depends on the material of which the string is composed and is seen to be that force which would stretch the string to twice its natural length.

In the case of a compressed spring, provided the thrust lies within a certain limit, a similar law holds, so that the force causing compression is proportional to the amount of compression below the natural length of the spring.

The extension of a heavy elastic string (weight  $w$  and natural length  $l$ ) hanging from one end and supporting a weight ( $W$ ) at the other is determined as follows.

Let  $OB$  represent the string when unstretched and  $O'B'$  (length  $L$ ) when stretched.

Suppose the element  $PQ$  of length  $\delta x$  becomes  $P'Q'$  of length  $\delta x'$  when stretched.

$$\text{Then } \lambda \frac{\delta x' - \delta x}{\delta x} = W + \frac{w}{l}(l - x)$$

$$\therefore \delta x' = \left[ 1 + \frac{W}{\lambda} + \frac{w}{\lambda l}(l - x) \right] \delta x$$

$$\therefore \int_0^L \delta x' = \int_0^l \left[ 1 + \frac{W}{\lambda} + \frac{w}{\lambda l}(l - x) \right] dx$$

$$\therefore L = l + \frac{W}{\lambda}l + \frac{w}{\lambda}l - \frac{w}{2\lambda}l$$

$$= l + \frac{l}{\lambda} \left( W + \frac{w}{2} \right).$$

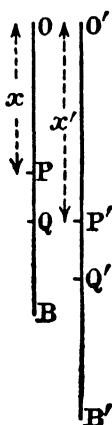
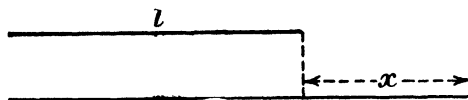


Fig. 11.

The extension is therefore  $\frac{l}{\lambda} \left( W + \frac{w}{2} \right)$ .

If the string be light the extension is  $\frac{l}{\lambda} W$  and if it be heavy and hanging under its own weight alone, the extension is  $\frac{l}{\lambda} \cdot \frac{w}{2}$ . Thus the extension due to the weight of a heavy string is the same as if the string were light and a weight equal to half the weight of the string were supported at its lower extremity.



The work done in stretching an elastic string from its natural length  $l$  to a length

$l + x$  is  $\int_0^x T dy$  where  $T$  is the tension when the extension is  $y$ .

Now  $T = \frac{\lambda}{l} y \therefore$  the work done is  $\frac{\lambda}{l} \int_0^x y dy$ , i.e.  $\frac{\lambda x^2}{2l}$  or

$\frac{\lambda}{l} \cdot \frac{x}{2} \times x$ . The energy of a stretched string is therefore given

by half the product of the tension and the extension.

**15. Newton's Law of Attraction.**—This law states that between every two particles of masses  $m$  and  $m'$  at a distance  $d$  apart, the mutual attraction is

$$k \frac{mm'}{d^2}$$

units of force, where  $k$  is a constant.

If the masses be measured in pounds and the length in feet the value of  $k$  is  $1.075 \times 10^{-9}$  approximately and the attraction is expressed in poundals. If the masses and length be expressed in C.G.S. units, the above formula gives the attraction in dynes when  $k = 6.66 \times 10^{-8}$  approximately.

It is proved in a subsequent chapter that in the case of spheres, the above formula holds if  $m$  and  $m'$  are the masses respectively and  $d$  is the distance apart of their centres.

Modern investigations indicate that Newton's Law of Attraction is not absolutely true but the error involved by its application is extremely small.

**16. Units.**—The fundamental units are Time (T), Length (L) and Mass (M). Other units employed in Mechanics are derived from these and it is useful to know their dimensions. An angle, being measured by the ratio of arc subtended to

radius, has no dimensions. A list of dimensions is given as follows :—

| <i>Unit.</i>                          | <i>Dimensions.</i> |
|---------------------------------------|--------------------|
| Area ... ..                           | $L^2$              |
| Volume ... ..                         | $L^3$              |
| Linear Velocity ... ..                | $L.T^{-1}$         |
| Angular Velocity ... ..               | $T^{-1}$           |
| Linear Acceleration... ..             | $L.T^{-2}$         |
| Angular Acceleration ... ..           | $T^{-2}$           |
| Force ... ..                          | $M.L.T^{-2}$       |
| Momentum ... ..                       | $M.L.T^{-1}$       |
| Impulse ... ..                        | $M.L.T^{-1}$       |
| Turning Moment ... ..                 | $M.L^2.T^{-2}$     |
| Density (mass per unit Volume) ... .. | $M.L^{-3}$         |
| Pressure (Force per unit Area) ... .. | $M.L^{-1}.T^{-2}$  |
| Energy or Work ... ..                 | $M.L^2.T^{-2}$     |
| Power (Work per unit Time) ... ..     | $M.L^2.T^{-3}$     |

If the fundamental units be changed, any new derived unit can be readily expressed in terms of the corresponding original derived unit by the method indicated by the following example :—

Taking the units of mass, time and length to be 1 ounce, 1 minute and 1 mile respectively, express the new unit of force in terms of poundals.

The dimensions of force are  $M.L.T^{-2}$ .

$$\begin{aligned} \therefore 1 \text{ new unit of force} &= \frac{1}{16} \times 5280 \times (60)^{-2} \text{ poundal} \\ &= \frac{11}{120} \text{ poundal.} \end{aligned}$$

In any equation expressing the relation between distance, velocity, force, etc., the dimensions of each term must be the same. For example, in the case of a particle moving with uniform acceleration, we have the relation

$$s = ut + \frac{1}{2} at^2.$$

The dimension of  $s$  is of course  $L$ .

„ „ „  $ut$  is  $L.T^{-1}.T$  i.e.  $L$ .

„ „ „  $at^2$  is  $L.T^{-2}.T^2$  i.e.  $L$ .

By consideration of dimensions many formulae, except for a numerical constant factor, can be obtained. For example, the kinetic energy ( $e$ ) of a particle can depend only on its mass ( $m$ ) and its velocity ( $v$ ).

Suppose that  $e = km^x v^y$  where  $k$  is a numerical constant. The dimensions of  $e$  are  $M.L^2.T^{-2}$  and those of  $m^x v^y$  are  $M^x L^y T^{-y}$ . Equating these dimensions,  $x = 1$  and  $y = 2$  so that  $e = kmv^2$ .

Again, the time ( $t$ ) of oscillation of a simple pendulum could only depend on the mass ( $m$ ) of the bob, the length ( $l$ ) of the string and the acceleration due to gravity ( $g$ ).

Let  $t = km^2 v g^s$  where  $k$  is a numerical constant.

Then equating dimensions,

$$T = M^x \cdot L^y \cdot L^z T^{-u}$$

so that  $x = 0, y + z = 0$  and  $-2z = 1$

$\therefore z = -\frac{1}{2}$  and  $y = \frac{1}{2}$ .

Thus

$$t = k \sqrt{\frac{l}{g}}.$$

### 17. Simple Pendulum.—

Let  $m \equiv$  mass of the bob (particle).

Let  $l \equiv$  length of string (light).

A is the lowest position of the bob and B is the position at time  $t$  from A.

Let arc AB =  $s$  and  $\widehat{AOB} = \theta$ .

Apply the equation of motion along the tangent at B,

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta.$$

For small oscillations,  $\theta$  is always small and

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

The equation of motion therefore for small oscillations is

$$\frac{d^2 s}{dt^2} = -g \theta.$$

Now

$$s = l\theta$$

$\therefore$

$$\frac{d^2 s}{dt^2} = -\frac{g}{l} s.$$

As  $\frac{g}{l}$  is constant, this is an equation of Simple Harmonic

Motion and the periodic time is  $\frac{2\pi}{\sqrt{\frac{g}{l}}}$  i.e.  $2\pi \sqrt{\frac{l}{g}}$ .

If a particle move with S.H.M. determined by the equation  $\frac{d^2 x}{dt^2} = -\mu x$ , the length ( $l$ ) of the simple equivalent pendulum

is given by  $\mu = \frac{g}{l}$  i.e.  $l = \frac{g}{\mu}$ .

**Ex. I.**—A light elastic string whose natural length is 4 ft. is such that a force of 12 lb. wt. will stretch it to twice its natural length. One end of the string is fixed and a mass of 4 lb. is attached to the

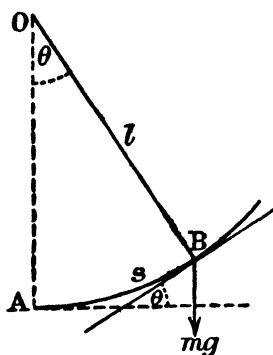


Fig. 13.

other. Initially the string is vertical and the mass is held so that the string is just unstretched. The mass is then let go: prove that its subsequent oscillation is simple harmonic and find the amplitude, periodic time, maximum velocity and length of the simple equivalent pendulum, taking  $g$  to be 32 ft. per sec. per sec.

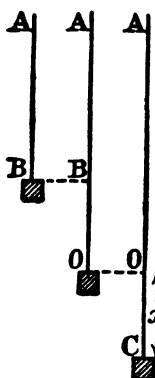


Fig. 14

Let AB represent the natural length.

Let AO represent the length if the mass were hanging in equilibrium.

Then by Hooke's law,

$$4g = 12g \cdot \frac{BO}{4}$$

$$\therefore BO = \frac{4}{3} \text{ ft. and thus } AO = 5\frac{1}{3} \text{ ft.}$$

Suppose that  $t$  sec. after passing position O, the mass is at C. Let  $OC = x$  ft. and let  $T$  pdl. be the tension of the string.

$$\text{Then } 4g - T = 4 \frac{d^2x}{dt^2} \dots \dots \dots (1)$$

$$\text{and } T = 12g \cdot \frac{x + 1\frac{1}{3}}{4} \dots \dots \dots (2)$$

$$\therefore 4g - 3g(x + 1\frac{1}{3}) = 4 \frac{d^2x}{dt^2}$$

$$\text{i.e. } \frac{d^2x}{dt^2} = -\frac{3g}{4} \cdot x.$$

$\therefore$  the motion is simple harmonic, the periodic time being

$$\frac{2\pi}{\sqrt{\frac{3g}{4}}} \text{ sec.} = \frac{\sqrt{6}\pi}{6} \text{ sec.}$$

Let  $v$  ft. per sec. be the velocity of the mass at distance  $x$  below O  
then  $v \frac{dv}{dx} = -\frac{3g}{4} x$ ;

$$\therefore v^2 = -\frac{3g}{4} x^2 + k, \text{ where } k \text{ is a constant.}$$

$$\text{When } x = -\frac{4}{3}, v = 0, \therefore k = \frac{4g}{3}$$

$$\therefore v^2 = -\frac{3g}{4} x^2 + \frac{4g}{3}.$$

When  $v = 0$ ,  $x = \pm \frac{4}{3}$  and thus O is the centre of oscillation and the amplitude is therefore  $1\frac{1}{3}$  ft.

Again as  $-\frac{3g}{4}x^2$  is always negative, the maximum velocity is  $\frac{8}{3}\sqrt{6}$  ft. per sec.

Let  $l$  ft. be the length of the simple equivalent pendulum ; then  $\frac{3g}{4} = \frac{g}{l}$  and thus the length is  $1\frac{1}{3}$  ft.

**Ex. II.** A uniform flexible chain 8 ft. long rests in a straight line on a smooth horizontal table except for a portion 1 ft. long which hangs over an edge at right angles to the chain. If the chain moves from rest, calculate the time taken to leave the table. Also calculate the time if the coefficient of friction be .125 ( $g = 32$  ft. per sec. per sec.).

Let  $x$  ft. be the length over the edge and  $v$  ft. per sec. the velocity of the chain at the end of  $t$  sec.

$$\text{Then} \quad 8v \frac{dv}{dx} = 32x$$

$$\therefore \quad v^2 = 4x^2 + k, \text{ where } k \text{ is a constant.}$$

$$\text{When} \quad v = 0, x = 1 \quad \text{so that } k = -4$$

$$\therefore \quad v^2 = 4(x^2 - 1).$$

$$\therefore \quad \frac{dx}{dt} = 2\sqrt{x^2 - 1}, \text{ the velocity being in the } +ve \text{ direction}$$

$$\therefore \quad \frac{1}{2} \int \frac{dx}{\sqrt{x^2 - 1}} = \int dt$$

$$\therefore \quad \frac{1}{2} \log_e(x + \sqrt{x^2 - 1}) = t + k_1 \text{ where } k_1 \text{ is a constant.}$$

$$\text{When} \quad t = 0, x = 1 \quad \text{so that } k_1 = \frac{1}{2} \log_e 1 = 0$$

$$\therefore \quad t = \frac{1}{2} \log_e(x + \sqrt{x^2 - 1}).$$

$$\therefore \text{ time taken for the chain to leave the table} \\ = \frac{1}{2} \log_e(8 + \sqrt{63}) \text{ sec.}$$

When friction is taken into account, the equation of motion is

$$8v \frac{dv}{dx} = 32x - .125 \times 32(8 - x)$$

$$\text{i.e.} \quad v \frac{dv}{dx} = 4.5x - 4.$$

$$\therefore \quad \int v dv = \int (4.5x - 4) dx$$

$$\therefore \quad v^2 = 4.5x^2 - 8x + k_2, \text{ where } k_2 \text{ is a constant}$$

$$\text{When} \quad v = 0, x = 1 \text{ so that } k_2 = 8 - 4.5 = 3.5$$

$$\therefore \quad v^2 = \frac{1}{2}(9x^2 - 16x + 7)$$

$$\therefore \quad \sqrt{2} \int \frac{dx}{\sqrt{9x^2 - 16x + 7}} = \int dt.$$

$$\begin{aligned}\text{Now } \sqrt{9x^2 - 16x + 7} &= 3 \sqrt{x^2 - \frac{16}{9}x + \frac{64}{81} + \frac{7}{9} - \frac{64}{81}} \\ &= 3 \sqrt{\left(x - \frac{8}{9}\right)^2 - \frac{1}{81}}\end{aligned}$$

$$\therefore \frac{\sqrt{2}}{3} \int \frac{dx}{\sqrt{\left(x - \frac{8}{9}\right)^2 - \frac{1}{81}}} = \int dt$$

$$\therefore \frac{\sqrt{2}}{3} \log_e \frac{\left(x - \frac{8}{9}\right) + \sqrt{\left(x - \frac{8}{9}\right)^2 - \frac{1}{81}}}{\frac{1}{9}} = t + k_2, \text{ where}$$

$k_2$  is a constant.

$$\text{When } t = 0, x = 1 \text{ so that } k_2 = \frac{\sqrt{2}}{3} \log_e 1 = 0$$

$$\therefore t = \frac{\sqrt{2}}{3} \log_e (9x - 8 + 3\sqrt{9x^2 - 16x + 7})$$

$$\therefore \text{time taken for chain to leave the table} = \frac{\sqrt{2}}{3} \log_e (64 + 3\sqrt{455}) \text{ sec.}$$

**Ex. III.** If the Earth were still, calculate the least velocity with which a shot should be projected vertically upwards in order that it should never return. Assume the earth's radius to be 4000 miles.

Let  $M, m$  lb. be the mass of the earth and the particle respectively.

Let  $r$  ft. be the radius of the earth.

Let  $v$  ft. per sec. be the velocity of the particle when at a distance  $x$  ft. from the point of projection.

$$\text{Then } mv \frac{dv}{dx} = - \frac{k m M}{(x + r)^2}, \text{ where } k \text{ is a constant.}$$

$$\therefore \int v dv = - k M \int \frac{dx}{(x + r)^2}$$

$$\text{i.e., } \frac{v^2}{2} = \frac{k M}{x + r} + C \text{ (a constant).}$$

From the condition that the velocity of projection is the least possible, when  $x = \infty, v = 0$  so that  $C = 0$ .

Let  $V$  ft. per second be the velocity when  $x = 0$ ,

$$\text{then } V^2 = \frac{2 k M}{r}.$$

Considering a mass of 1 lb. at the earth's surface, we have

$$\frac{k M}{r^2} = g$$



$$\begin{aligned} \therefore V^2 &= 2gr \\ \therefore \text{least velocity of projectile} &= \sqrt{64 \times 4000 \times 5280} \text{ ft./sec.} \\ &= \frac{8 \times 200 \times \sqrt{528}}{5280} \text{ ml./sec.} \\ &= 7.0 \text{ ml./sec.} \end{aligned}$$

**Ex. IV.** An engine working with constant horse-power  $H$  can pull a train of  $M$  tons on the level against a resistance of  $R$  lb. wt. per ton, at  $V$  miles per hour. Find the initial acceleration when it starts to pull the same train from rest. Find an expression also for the time taken to reach (i.) a vel. of  $V_1$  miles per hour; (ii.) its maximum speed.

Work done per sec. against resistance when the train is moving with uniform velocity  $\frac{22V}{15}$  ft./sec.  $= \frac{22RMV}{15}$  ft. lb.

$$\therefore \frac{22RMV}{15} = 550H \quad \dots \quad \dots \quad \dots \quad (1)$$

Let  $F$  lb. wt. be the tractive force when the velocity is  $v$  ft./sec.  
then  $Fv = 550H \quad \dots \quad \dots \quad \dots \quad (2)$

Let the velocity be  $v$  ft./sec. after  $t$  sec. from rest, then

$$32(F - RM) = 2240M \cdot \frac{dv}{dt};$$

$$\therefore \text{ from (1) and (2) } RM \left[ \frac{22V}{15v} - 1 \right] = 70M \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = \frac{R}{70} \cdot \frac{22V - 15v}{15v}.$$

When  $v = 0$ ,  $\frac{dv}{dt} = \infty$ , i.e., the initial acceleration is infinite.

$$\begin{aligned} \frac{dt}{dv} &= \frac{70}{R} \cdot \frac{15v}{22V - 15v} \\ &= \frac{70}{R} \left[ \frac{22V}{22V - 15v} - 1 \right]; \end{aligned}$$

$$\therefore t = \frac{70}{R} \cdot \left[ -\frac{22V}{15} \log_e (22V - 15v) - v \right] + C.$$

When  $v = 0$ ,  $t = 0$  so that  $C = \frac{70}{R} \cdot \frac{22V}{15} \log_e 22V$ .

$$\therefore t = \frac{70}{R} \left[ \frac{22V}{15} \log_e \frac{22V}{22V - 15v} - v \right].$$

The time taken to reach a velocity of  $V_1$  ml. per hr. is obtained by putting  $v = \frac{22V_1}{15}$  and is therefore  $\frac{308}{3R} \left[ V \log_e \frac{V}{V - V_1} - V_1 \right]$ .

The maximum velocity is  $V$  ml. per hr. and when  $V_1 = V$  the time taken from rest is infinite.

NOTE 1.—Suppose an engine to be exerting at any instant a force of  $F$  lb. wt. and that the train moves through a distance  $\delta s$  ft. in  $\delta t$  sec. Then the work done by the engine in  $\delta t$  sec. is  $F\delta s$  lb. wt. and thus the rate of doing work is  $F \frac{ds}{dt}$ , i.e.  $Fv$  ft. lb. per sec., where  $v$  ft. per sec. is the vel. of the train at the instant under consideration. Hence the horse power of the engine is determined by  $\frac{Fv}{550}$ . Suppose  $R$  lb. wt. to be the resistance, then the rate of doing work of the engine is  $Rv$  ft. lb. per sec. together with the rate of increase of kinetic energy of the train, i.e.  $\left(Rv + \frac{m}{g} v \frac{dv}{dt}\right)$  ft. lb. per sec. where  $m$  lb. is the mass of the train. These two results agree for considering the equation of motion,  $(F - R)g = m \frac{dv}{dt}$ , we have

$$Fv = Rv + \frac{m}{g} v \frac{dv}{dt}.$$

Ex. V. The total wind and friction resistance on a motor car is of the form  $(a + bv^2)$ . A car which weighs 3200 lb. requires 10 H.P. (effective) at 30 miles per hour and 6 H.P. at  $22\frac{1}{2}$  miles per hour. Find in what distance the speed will fall from 30 to 15 miles per hour when coasting on a level road with the clutch out.

When the car is travelling at uniform speed  $v$  ft. per sec. the horse power is  $\frac{(a + bv^2)v}{550}$ ,  $(a + bv^2)$  being measured in lb. wt.

$$\begin{aligned} \therefore \quad & 44a + 44^3b = 550 \times 10 \\ \text{i.e.,} \quad & a + 44^2b = 125 \quad \dots \quad \dots \quad \dots \quad (1) \\ \text{and} \quad & a + 33^2b = 100 \quad \dots \quad \dots \quad \dots \quad (2) \end{aligned}$$

When the clutch is out, the equation of motion is

$$3200 \frac{v dv}{dx} = -32(a + bv^2),$$

where  $x$  is the distance travelled in ft. from the instant the velocity is 44 ft. per sec. to the instant when it is  $v$  ft. per sec.

$$\begin{aligned} \therefore \quad & \int \frac{v dv}{a + bv^2} = -\frac{1}{100} \int dx \\ \therefore \quad & \frac{1}{2b} \log_e (a + bv^2) = -\frac{x}{100} + C \text{ (a constant).} \end{aligned}$$

When  $x = 0$ ,  $v = 44$  so that  $C = \frac{1}{2b} \log_e (a + 44^2b)$

$$\therefore \quad \frac{x}{100} = \frac{1}{2b} \log_e \frac{a + 44^2b}{a + bv^2}.$$

$\therefore$  distance travelled when the velocity is 22 ft. per sec. is

$$\frac{100}{2b} \log_e \frac{a + 44^2b}{a + 22^2b} \text{ ft.}$$

From (1) and (2),  $b = \frac{25}{77 \times 11}$ ,

$$a + 44^3 b = 125$$

and  $a + 22^3 b = 100 - 55 \times 11b = 100 - \frac{125}{7} = \frac{575}{7}$

$\therefore$  distance travelled  $= 77 \times 22 \times \log_e \frac{35}{23}$  ft.

**Ex. VI.** Taking as rectangular co-ordinates the kinetic energy of a particle moving in a straight line and the reciprocal of the resultant force acting on it, and drawing a curve to represent the motion, show that the distance travelled is represented by the area under this curve.

Sketch the curve for the case of a vehicle driven by an engine which is working at constant horse-power, the only resistance being a force proportional to the square of the velocity. Show that the velocity has a certain limiting value; and calculate the distance travelled while the velocity increases from a half to three-quarters of this limiting value, taking the mass of the vehicle as 2000 pounds, the horse power as 100, and the limiting velocity as 150 feet per second.

Using the ordinary notation, we have

$$x = \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 \text{ and } y = \frac{1}{F} = \frac{1}{m \frac{d^2 s}{dt^2}}$$

$$\begin{aligned} \therefore y dx &= \frac{1}{m \frac{d^2 s}{dt^2}} \cdot m \left( \frac{ds}{dt} \right) \cdot \frac{d^2 s}{dt^2} \cdot dt \\ &= ds \end{aligned}$$

$$\therefore \int y dx = s.$$

Let  $P$  be the tractive force on the vehicle, then

$$x = \frac{1}{2} mv^2 \text{ and } y = \frac{1}{P - kv^2} \text{ where } k \text{ is a constant;}$$

but  $P \cdot v = C$  (a constant)  $\therefore y = \frac{v}{C - kv^3}$ .

Thus, giving  $v$  different values, sets of corresponding values of  $x$  and  $y$  are obtained by means of which the curve connecting  $x$  and  $y$  can be drawn. (This is left as an exercise for the student.)

The maximum velocity is obtained when the resultant force on the vehicle is zero, i.e. when  $P = kv^2$ , i.e. when  $kv^2 \cdot v = C$

$$\therefore \text{the maximum velocity is } \sqrt[3]{\frac{C}{k}}.$$

Using the given numerical values,

$$x = 1000v^2, C = 55000 \times 32 \text{ and } \frac{C}{k} = 150 \times 150 \times 150$$

$$\therefore k = \frac{11 \times 32}{3 \times 15 \times 15} \text{ and } y = \frac{v}{32 \left( 55000 - \frac{11v^3}{45 \times 15} \right)}.$$

$$\begin{aligned} \text{Distance travelled } \left( \int y dx \right) &= \int_{\frac{3 \times 75}{75}}^{\frac{3 \times 75}{2}} \frac{2000 v^3 dv}{32 \left( 55000 - \frac{11v^3}{45 \times 15} \right)} \text{ ft.} \\ &= -\frac{125}{2} \cdot \frac{45 \times 15}{3 \times 11} \log_e \frac{55000 - \frac{11 \times 27 \times 75^3}{45 \times 15 \times 8}}{55000 - \frac{11 \times 75^3}{45 \times 15}} \text{ ft.} \\ &= \frac{125 \times 15^3}{22} \log_e \frac{48125}{31796\frac{1}{8}} \text{ ft.} \end{aligned}$$

**Ex. VII.** AB is a straight horizontal smooth wire. A bead D slides along the wire and is at A when a string DCB is horizontal. At C, another bead of equal mass as D is fixed to the string and the end B is fixed to the wire. DC = CB =  $a$ . Determine the velocity and acceleration of D when it is at distance  $a$  from A.

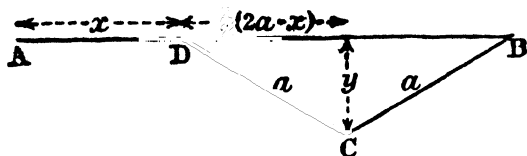


Fig. 15.

At the end of time  $t$ , let  $AD = x$  and let the distance of C below  $AB = y$

Work done by gravity on the system =  $mgy$

$$\therefore mgy = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} m \left[ \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \right] \quad \dots (1)$$

Now

$$y = \frac{1}{2} \sqrt{4ax - x^2}$$

$$\therefore \frac{dy}{dt} = \frac{2a - x}{2\sqrt{4ax - x^2}} \cdot \frac{dx}{dt}.$$

Subst. in (1), then

$$g\sqrt{4ax - x^2} = \left( \frac{dx}{dt} \right)^2 \left[ \frac{5}{4} + \frac{4a^2 - 4ax + x^2}{4(4ax - x^2)} \right]$$

$$\therefore \left( \frac{dx}{dt} \right)^2 = \frac{g(4ax - x^2)^{\frac{3}{2}}}{a^2 + 4ax - x^2} \quad \dots \dots \dots (2)$$

Differentiating with respect to  $t$ , then

$$\begin{aligned} &2 \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} \\ &= g \cdot \frac{(a^2 + 4ax - x^2)^{\frac{1}{2}} \left[ \frac{1}{2} (4ax - x^2)^{\frac{1}{2}} (4a - 2x) \right] - (4ax - x^2)^{\frac{3}{2}} (4a - 2x)}{(a^2 + 4ax - x^2)^3} \cdot \frac{dx}{dt} \end{aligned}$$

$$\therefore \frac{d^2x}{dt^2} = \frac{g}{2} \frac{(2a-x)(4ax-x^2)^{\frac{1}{2}}(3a^2+4ax-x^2)}{(a^2+4ax-x^2)^2} \quad \dots (3)$$

From (2), the velocity when  $x = a$  is  $\left[ \frac{3\sqrt{3ga}}{4} \right]^{\frac{1}{2}}$

From (3), the acceleration when  $x = a$  is  $\frac{3\sqrt{3}g}{16}$ .

NOTE 2.—By applying to the whole system the principle that work done equals kinetic energy produced, the consideration of the tensions of the portions of the string is avoided.

**Ex. VIII.** Two spheres, each of diameter  $d$ , have masses  $M$  and  $m$  respectively. Initially they are at rest with their centres a distance  $a$  apart and are free to move under mutual attraction  $\frac{k m M}{(\text{distance})^2}$ . Find the distance apart of the centre of the spheres after  $n$  impacts, the co-efficient of restitution being  $e$ .



Fig. 16.

Let  $A, B$  be the initial positions of the centres,  $A_1, B_1$ , the positions after time  $t$ , and  $A_2, B_2$  the positions when the spheres touch.

Let  $G$  be the centre of inertia of the spheres, then as the linear momentum of the system is always zero,  $G$  is a fixed point.

$$\text{Let } A_1G = x, \text{ then } A_1B_1 = \frac{(M+m)x}{m} \text{ and } B_1G = \frac{Mx}{m}.$$

$$\text{Also } AG = \frac{ma}{M+m}, BG = \frac{Ma}{M+m}, A_2G = \frac{md}{M+m}, B_2G = \frac{Md}{M+m}.$$

Let  $v$  be the velocity of the mass  $M$  at  $A_1$ ,

$$\text{then} \quad -Mv \frac{dv}{dx} = \frac{km^3M}{(M+m)^2x^3}$$

$$\therefore \frac{v^2}{2} = \frac{km^3}{(M+m)^2} \cdot \frac{1}{x} + C, \text{ a constant.} \quad \dots \dots (1)$$

$$\text{When } x = \frac{ma}{M+m}, v = 0 \text{ so that } C = -\frac{km^2}{a(M+m)}$$

$$\therefore v^2 = \frac{2km^3}{M+m} \left[ \frac{m}{M+m} \cdot \frac{1}{x} - \frac{1}{a} \right].$$

Let  $U_1$  be the vel. of mass  $M$  just before the first impact,

$$\text{then} \quad U_1 = m \cdot \sqrt{\frac{2k}{M+m} \left[ \frac{1}{d} - \frac{1}{a} \right]^{\frac{1}{2}}}.$$

Similarly, if  $U_2$  be the vel. of mass  $m$  just before the first impact, then

$$U_2 = M \cdot \sqrt{\frac{2k}{M+m}} \left[ \frac{1}{d} - \frac{1}{a} \right]^{\frac{1}{2}}.$$

Let  $V_1, V_2$  be the velocities immediately after the first impact, then

$$e(U_1 + U_2) = V_1 + V_2 = V_1 \left[ 1 + \frac{M}{m} \right] \text{ as } MV_1 = mV_2$$

$$\therefore V_1 = \frac{me(U_1 + U_2)}{M+m} \text{ and } V_2 = \frac{Me(U_1 + U_2)}{M+m}$$

$$\text{i.e. } V_1 = me \sqrt{\frac{2k}{M+m}} \left[ \frac{1}{d} - \frac{1}{a} \right]^{\frac{1}{2}} \text{ and } V_2 = Me \sqrt{\frac{2k}{M+m}} \left[ \frac{1}{d} - \frac{1}{a} \right]^{\frac{1}{2}}.$$

Eqn. (1) applies to the motion of the mass  $M$  after impact but the constant is different and is determined by the relation,  $x = \frac{md}{M+m}$

$$\text{when } v = V_1, \text{ and thus is } \frac{1}{2} \cdot m^2 \cdot e^2 \cdot \frac{2k}{M+m} \left[ \frac{1}{d} - \frac{1}{a} \right] - \frac{km^2}{M+m}.$$

The equation of motion after impact is therefore

$$\frac{v^2}{2} = \frac{km^2}{(M+m)^2} \cdot \frac{1}{x} + \frac{km^2}{M+m} \left[ e^2 \left( \frac{1}{d} - \frac{1}{a} \right) - \frac{1}{d} \right].$$

$$\text{When } v = 0, \frac{1}{x} = \frac{M+m}{m} \cdot \left[ \frac{a - e^2(a-d)}{ad} \right]$$

$\therefore$  the mass  $M$  after one impact, comes to rest at distance

$$\frac{m}{M+m} \cdot \left[ \frac{ad}{a - e^2(a-d)} \right] \text{ from G.}$$

As the linear momentum is zero, the mass  $m$  is at rest at the same instant, and as G is fixed the distance apart at this instant is

$$\frac{ad}{a - e^2(a-d)}.$$

Assume that the distance apart of the masses when at rest after  $n$  impacts is

$$\frac{ad}{a - e^{2n}(a-d)};$$

then when at rest after  $n+1$  impacts, the distance apart will be

$$\begin{aligned} & \frac{\frac{ad^2}{a - e^{2n}(a-d)}}{\frac{ad}{a - e^{2n}(a-d)} - e^2 \left\{ \frac{ad}{a - e^n(a-d)} - d \right\}} \\ &= \frac{ad^2}{ad - e^2 \{ ad - ad + e^{2n}d(a-d) \}} = \frac{ad}{a - e^{2(n+1)}(a-d)}. \end{aligned}$$

Now the distance apart after  $n$  impacts, namely  $\frac{ad}{a - e^{2n}(a-d)}$

has been proved to be true when  $n = 1$  and is therefore true when  $n = 2, 3, \dots n$ .

**Ex. IX.** An elastic string AB of length  $l$  is fixed at A and would be stretched to double its length if a weight  $W$  were fastened to B and gently lowered. A weight  $\frac{W}{10}$  is fastened to B and let fall from A.

Find the distance it will fall and prove that the period of the subsequent oscillation is  $\sqrt{\frac{l}{10g}} \left[ 4\sqrt{5} + \pi + 2 \sin^{-1} \frac{1}{\sqrt{21}} \right]$ .

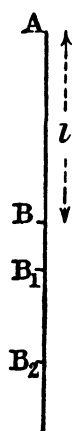


Fig 17

Let  $B_1$  be the position of mass  $\frac{W}{10g}$  when in equilibrium, then  $BB_1 = \frac{l}{10}$ . Let  $B_2$  be the lowest position, then denoting  $AB_2$  by  $y$ ,

$$\frac{W}{10}y = \frac{W}{2l}(y-l)^2$$

$$\therefore 5y^2 - 11yl + 5l^2 = 0$$

$$\therefore AB_2 = \frac{11 + \sqrt{21}}{10}l \quad \therefore B_1B_2 = \frac{\sqrt{21}}{10}l.$$

Let  $x$  be the distance below  $B_1$  of the mass after time  $t$  from the instant when it is at  $B_1$ ,

$$\text{then} \quad \frac{W}{10g} \cdot \frac{d^2x}{dt^2} = \frac{W}{10} - \frac{W}{l} \left( \frac{l}{10} + x \right)$$

$$\text{i.e.} \quad \frac{d^2x}{dt^2} = -\frac{10g}{l}x.$$

$\therefore$  the motion between B and  $B_2$  is simple harmonic, the centre of oscillation being  $B_1$  and the amplitude  $\frac{\sqrt{21}}{10}l$ .

$$\text{The time taken from } B_1 \text{ to } B_2 \text{ and back} = \pi \sqrt{\frac{l}{10g}}.$$

Let  $t_1$  be the time taken from  $B_1$  to B, then

$$\frac{l}{10} = \frac{\sqrt{21}}{10}l \sin \sqrt{\frac{10g}{l}} \cdot t_1$$

$$t_1 = \sqrt{\frac{l}{10g}} \sin^{-1} \frac{1}{\sqrt{21}}.$$

$$\text{The time taken from B to } B_1 \text{ is also } \sqrt{\frac{l}{10g}} \sin^{-1} \frac{1}{\sqrt{21}}.$$

Between A and B, the mass moves freely under gravity and thus the total time from A to B and from B to A is

$$2\sqrt{\frac{2l}{g}} \quad \text{i.e.,} \quad \sqrt{\frac{l}{10g}} \cdot 4\sqrt{5}.$$

The whole period of oscillation is therefore

$$\sqrt{\frac{l}{10g}} \cdot \left[ 4\sqrt{5} + \pi + 2 \sin^{-1} \frac{1}{\sqrt{21}} \right].$$

### Problems II

1. One end of an elastic spring is fixed at A, to the other end is fastened a particle heavy enough to stretch the spring to double its normal length  $a$ . Shew that if the particle is dropped from A, it will descend a distance  $(2 + \sqrt{3})a$  before coming to rest.

2. A horizontal plate with a particle resting on it is made to oscillate vertically with simple harmonic motion through a distance of 2 feet. What is the smallest number of vibrations per minute that will throw the particle off the plate?

3. A mass of 1 lb. is hung on a light spiral spring and produces a static deflection of  $1\frac{1}{2}$  inches. A mass of 1 lb. is suddenly added to the original mass. (i.) Find the maximum elongation produced; (ii.) show that the time of an oscillation of the whole mass is approximately  $\frac{1}{3}$  sec.

4. A heavy particle is attached by an elastic string to a fixed point on a smooth horizontal table. The modulus of elasticity of the string is equal to the weight of the particle and the particle is held on the table with the string stretched to twice its natural length ( $a$ ). The particle is let go; shew that it will return to the starting point after a time  $2(\pi + 2)\sqrt{\frac{a}{g}}$ .

5. A particle when hanging in equilibrium at the end of a light elastic string stretches it a distance  $a$ . Prove that the period of vibration of the particle in a vertical line through its equilibrium position is the same as that of a simple pendulum of length  $a$ .

6. A mass of 10 lb. hanging at the end P of a light spiral spring produces an extension of 4 inches. A mass of 5 lb. is suddenly added at P; find how far P will fall before coming to rest; and shew that it will return to its initial position in a little less than  $\frac{1}{3}$ ths of a second.

7. A spring of negligible inertia carries a pan weighing 1 ounce, and is such that a  $\frac{1}{2}$  lb. weight will lower the pan by 1 inch. It is compressed 2 inches and placed on a table with its axis vertical: a 2 ounce weight is put on it and the spring released. Find how high the weight rises before it leaves the spring and its velocity at that instant.



8. A ring of mass  $m$  slides on a smooth vertical rod; attached to the ring is a light string passing over a smooth peg distant  $a$  from the rod, and at the other end of the string is a mass  $M$  ( $> m$ ). The ring is held on a level with the peg and released: show that it first comes to rest after falling a distance  $\frac{2mMa}{M^2 - m^2}$ .

9. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position, measured in the same direction, are 1, 5 and 5 ft. Show that the period of the complete oscillation is  $\frac{2\pi}{\cos^{-1}\frac{1}{5}}$  sec.

10. A body, weight  $W$ , moves in a straight line under a force, always directed to a fixed point  $O$  in the straight line, equal to  $kx$ , where  $x$  is the distance of the body from  $O$  at any moment. Prove that the work done on the body as it alters its distance from  $x_1$  to  $x_2$  is  $\frac{1}{2}k(x_1^2 - x_2^2)$  and that, if its greatest velocity is  $V$ , the body moves to and fro through a distance  $2V\sqrt{\frac{W}{gk}}$ .

11. A train weighing 420 tons runs from rest up an incline of 1 in 210 under a pull which varies with the distance as given in the following table. If the frictional resistance to the motion of the train is 16 lbs. per ton, find the velocity of the train after the first 800 yds.

|                 |    |      |     |     |     |     |     |     |     |
|-----------------|----|------|-----|-----|-----|-----|-----|-----|-----|
| Distance, yards | 0  | 100  | 200 | 300 | 400 | 500 | 600 | 700 | 800 |
| Force, tons wt. | 10 | 10.3 | 11  | 10  | 9   | 7.5 | 6.3 | 5.5 | 5   |

12. A horizontal board is made to perform simple harmonic oscillations horizontally, moving to and fro through a distance 30 inches and making 15 complete oscillations per minute. Find the least value of the coefficient of friction in order that a heavy body placed on the board may not slip.

13. A light endless elastic string of unstretched length  $2b$  passes over two small smooth pegs on the same level distant  $b$  apart. A particle is attached to a point on the string and when the particle is in equilibrium the string forms the three sides of an equilateral triangle. Prove that the period of vibration of the particle in a vertical line is the same as that of a pendulum of length  $\frac{2\sqrt{3}}{7}b$ .

14. A balloon ascends with constant acceleration, and reaches a height of 900 feet in one minute. Show that a pendulum clock carried in it will gain at the rate of about 28 seconds per hour.

15. Prove that the free oscillations of the mercury of a barometer in a U tube of uniform section will synchronize with a pendulum of half the length of the mercury column.

16. A uniform chain 20 feet long passes over a small smooth peg and initially the length of one portion is 11 feet. If allowed to move from rest, calculate the time that elapses before the chain leaves the peg.

17. A light spring is kept compressed by the action of a given force; the force is suddenly reversed; prove that the greatest subsequent extension of the spring is three times its initial contraction.

18. One end of an elastic string, whose modulus of elasticity is  $\lambda$  and whose unstretched length is  $a$ , is fixed to a point on a smooth horizontal table and the other end is tied to a particle of mass  $m$  which is lying on the table. The particle is pulled to a distance where the extension of the string is  $b$ , and then let go; show that

the time of a complete oscillation is  $2\left(\pi + \frac{2a}{b}\right)\sqrt{\frac{am}{\lambda}}$ .

19. A particle is attached to a point in a rough plane inclined at an angle  $\alpha$  to the horizon; originally the string was unstretched and lay along a line of greatest slope; show that the particle will oscillate only if the co-efficient of friction is  $< \frac{1}{2} \tan \alpha$ .

20. A train can be accelerated by a force of 55 lb. per ton weight, and when steam is shut off can be braked by a force of 440 lb. per ton weight. Find the least time between stopping stations 3850 feet apart, the greatest velocity of the train and the horse power per ton weight of train necessary for the engine. [Assume  $g = 32$  ft. per sec. per sec.]

21. A string AB consists of two portions AC, CB, of unequal lengths and elasticities. The composite string is stretched and held in a vertical position with the ends A and B secured. A particle is attached to C, and the steady displacement of C is found to be  $d$ . Shew that a further small vertical displacement of C will cause the particle to execute a Simple Harmonic Motion, and that the length of the simple equivalent pendulum is  $d$ . Both portions of the string are assumed to be in tension throughout, and the weight of the string may be neglected.

22. The resistance to a bicycle, on a level road, expressed in lb. wt. is  $.01V^2 + .013W$ , where  $V$  is the velocity relative to the air, in miles per hour, and  $W$  is the weight of the machine and rider, in pounds. A cyclist who can only work at  $\frac{1}{10}$  of a horse-power, and whose slowest rate of riding is 3 miles an hour, tries to ride up a hill against a wind blowing at 7 miles per hour. Show that the hill will be too steep for him if its gradient exceeds 1 in 22.5 approximately and if  $W = 200$ .

23. A particle of mass  $m$  is placed in a smooth tube of mass  $M$ , which is closed at both ends and lies on a smooth horizontal table. The particle is projected from one end of the tube with velocity  $u$  and proceeds to rebound from each end alternately, the coefficient of resilience being  $e$ . Prove that the velocity of the tube is  $\frac{mu(1 + e^{2n-1})}{m + M}$  after  $(2n - 1)$  impacts and  $\frac{mu(1 - e^{2n})}{m + M}$  after  $2n$  impacts.

24. The friction resistance to the motion of a train, mass  $M$  is constant ( $R$ ), and the air-resistance ( $A$ ) is proportional to the

speed ( $v$ ). If the pull of the engine is constant ( $P$ ), find an expression for the time taken from rest to reach a speed  $V$ , on the level. If  $R = 10$  lb. wt. per ton,  $M = 100$  tons,  $A = 10$  lb. wt. per ton at 15 ml. per hr., and  $P = 3000$  lb. wt., find the time to reach 30 ml. per hr.

25. The resistances to the motion of a train are of two kinds, (a) friction (constant), (b) air resistance which varies as the speed. Express by equation, the relation between (a) pull of engine and maximum speed, (b) horse-power exerted and maximum speed. If the train be 150 tons, including the engine, and the resistances are (a) 4 lb. wt. per ton (b) 300 lb. wt. at 15 ml. per hr., calculate the maximum speed for a 200 H.P. engine. In what time will the train come to rest if steam is shut off when it is travelling 30 ml. per hr.?

26. A train starting from rest is uniformly accelerated until its velocity is 30 feet per second and then uniformly accelerated at half the previous rate until its velocity is 60 feet per second and after that the velocity remains uniform. The train takes 118 seconds to travel the first mile. Find the initial acceleration.

If the train weighs 200 tons and the resistances to motion are equivalent to a back pull of 16 lb. wt. per ton find the average horse-power and also the maximum horse-power at which the engine was working during the time it took to travel the first mile.

27. If the Moon and Earth were at rest, show that the least velocity with which a particle could be projected from the Moon, in order to reach the Earth, is about  $1\frac{1}{2}$  miles per second, assuming their radii to be 1100 and 4000 miles respectively, the distance between their centres 240,000 miles, and the mass of the Moon to be  $\frac{1}{81}$  that of the Earth.

28. A particle of mass ( $m$ ) lying on a smooth horizontal table is attached to two elastic strings whose natural lengths are ( $l$ ) and ( $l_1$ ), and moduli ( $\lambda$ ) and ( $\lambda_1$ ) respectively. The other ends of the strings are fixed to two points on the table at a distance apart greater than  $l + l_1$ . Show that if the particle vibrates in the line of the strings, its period will be

$$2\pi \sqrt{\frac{m}{\left(\frac{\lambda}{l} + \frac{\lambda_1}{l_1}\right)}}$$

29. A particle is suspended from a fixed point by an elastic string of unstretched length ( $a$ ). In the position of equilibrium the extension of the string is ( $e_1$ ). The particle is drawn down a further distance ( $e_2$ ) and freed. Determine the motion completely in the cases when (1)  $e_2 < e_1$ , (2)  $e_1 < e_2 < \sqrt{e_1(e_1 + 4a)}$ , (3)  $e_2 > \sqrt{e_1(e_1 + 4a)}$ .

30. A Tube railway train, weighing 200 tons, has four 125 horse-power motors; the train starts from rest down an incline of 1 in 100, the motors exerting at starting a pull of 25,000 pounds weight. Find the acceleration at starting and the full speed on the level, assuming the frictional resistances to be 30 lb. wt. per ton.

31. If the resistance to the motion of a body is proportional to the speed and is  $\cdot 01$  of its weight at 10 ft. per sec., find in what time its speed will be reduced from 100 ft. per sec. to 50 ft. per sec. If the body is set sliding down an incline of 1 vertical in 25 slope, what is the highest speed it can attain?

32. The tractive effort of an electric train is uniform and equal to the weight of 4 tons. The road resistance is 40 lbs. wt. per ton of the train, and the brake resistance is an additional 200 lb. wt. per ton. The train is taken from one station to the next, distant half-a-mile, in  $1\frac{1}{2}$  minutes, full power being kept on until the speed reaches 30 miles an hour, when the train "coasts" at a uniform speed until power is shut off and the brakes are put on. Show that the mass of the train is approximately 85 tons.

33. The resistance of the water to a ship's motion is proportional to the cube of the velocity; prove that, from the time the engines are stopped, the velocity  $v$  and distance run  $x$  are connected by an equation of the form  $\frac{dv}{dx} = -cv^2$ .

Deduce by integration, that if the velocity was  $u$  when the engines were stopped,  $\frac{1}{v} - \frac{1}{u} = cx$ .

Under such circumstances the speed of a ship of 5000 tons drops from 12 to 3 knots while she runs for 1 sea-mile. Find the horse-power necessary to propel her at 12 knots. A knot is a speed of 1 sea-mile i.e., 6080 feet, per hour.

34. A thin circular wire of radius  $a$  and mass  $m$  attracts a particle according to the law  $\frac{km_1m_2}{(\text{distance})^2}$ ; if a particle be placed on the axis at a small distance from the centre, find the time of a complete oscillation.

35. One end of a light spring, of natural length  $a$  and modulus  $mg$ , is fixed to a point in a smooth horizontal table, at a distance  $2a$  from its edge. To the other end is attached a particle of mass  $m$ , and to this particle is attached a light inextensible string which passes over the edge of the table and carries a mass  $2m$  at its other end. If the system is released from rest, with the spring at its natural length, the string at right angles to the edge of the table and the particle of mass  $2m$  hanging freely, show that the particle on the table will reach the edge after a time  $\pi(a/3g)^{\frac{1}{2}}$  with a speed  $(ag)^{\frac{1}{2}}$ .

36. A body of mass  $M$  lb. moves initially with a velocity of  $u$  ft. per sec. If a constant power equal to  $H$  horse-power be applied in the direction of its velocity, show that the time that elapses before the acceleration is reduced to  $\frac{1}{r}$ th of its original value is  $\frac{Mu^2(r^2-1)}{1100Hg}$  sec.

## CHAPTER III

### MOTION OF A PARTICLE IN ONE PLANE I

#### 18. Motion Referred to Fixed Axes at Right Angles.—

Let  $x$  and  $y$  be the co-ordinates of the particle at time  $t$ ; let  $v_x, v_y$  be the velocities and  $a_x, a_y$  the accelerations parallel to the axes OX and OY respectively at this time. Suppose one relation connecting any two of the quantities  $t, x, v_x$  and  $a_x$  be given as well as initial conditions, then the motion resolved along OX can, by Article 3, be completely determined. Similarly, given a relation between  $t, y, v_y$  and  $a_y$  and the necessary initial conditions, the motion resolved along OY can also be completely determined.

Suppose that  $x = f(t)$  and  $y = F(t)$ , then by eliminating  $t$  from these two equations, the equation to the path is obtained.

Also  $\frac{dx}{dt} = f'(t)$  and  $\frac{dy}{dt} = F'(t)$  so that the resultant velocity has magnitude  $\left[ \{f'(t)\}^2 + \{F'(t)\}^2 \right]^{\frac{1}{2}}$  and its direction is given by  $\tan^{-1} \frac{F'(t)}{f'(t)}$ . Similarly, the magnitude of the resultant acceleration is  $\left[ \{f''(t)\}^2 + \{F''(t)\}^2 \right]^{\frac{1}{2}}$  and its direction is given by  $\tan^{-1} \frac{F''(t)}{f''(t)}$ .

**Ex. I.** A particle moves with accelerations  $2t$  ft. per sec. per sec. parallel to OX and 6 ft. per sec. per sec. parallel to OY. Initially, the particle is at O and moving with velocity 8 ft. per sec. in the direction OX. Find the equation to the path and the positions of the particle when it moves in a direction equally inclined to the axes. Also find the resultant velocity when in these positions.

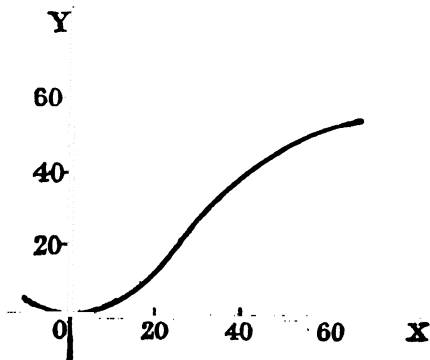


Fig. 18.

Let  $x, y$  be the co-ordinates after  $t$  sec.

Then  $\frac{d^2x}{dt^2} = 2t$  and  $\frac{d^2y}{dt^2} = 6$ .

$\therefore \frac{dx}{dt} = t^2 + 8$  and  $\frac{dy}{dt} = 6t$ .

$\therefore x = \frac{t^3}{3} + 8t$  and  $y = 3t^2$ .

Eliminating  $t$  from the last two equations, we find the equation of the path to be

$$243x^2 = y^3 + 144y^2 + 5184y.$$

Let  $\alpha$  be the angle its direction makes with OX after  $t$  sec. then

$$\tan \alpha = \frac{6t}{t^2 + 8};$$

$\therefore$  when equally inclined to the axes,  $t^2 - 6t + 8 = 0$

i.e.  $t = 2$  and  $4$ .

When  $t = 2$ ,  $x = 18\frac{2}{3}$ ,  $y = 12$ ,  $\frac{dx}{dt} = 12$  and  $\frac{dy}{dt} = 12$ .

Thus when its co-ordinates are  $18\frac{2}{3}$  ft. and 12 ft, the particle is moving at  $12\sqrt{2}$  ft. per sec. in a direction equally inclined to the axes.

Similarly when its co-ordinates are  $53\frac{1}{3}$  ft. and 48 ft., the particle is moving at  $24\sqrt{2}$  ft. per sec. in a direction equally inclined to the axes.

**Ex. II.** A particle describes a path with an acceleration which is always directed towards a fixed point O and varies directly as the distance from it. Initially the particle is on a line through O, at distance  $d$  from it, and moving with velocity  $V$  in a direction inclined at angle  $\alpha$  with the line. Determine the path.

Take O as the origin and the line as the axis of  $x$ . Let  $P(x, y)$  be the position of the particle after time  $t$ ; then resolving along the axes, we have

$$\frac{d^2x}{dt^2} = -\mu x \quad \text{and} \quad \frac{d^2y}{dt^2} = -\mu y$$

the most general solutions to which are:—

$$x = a \sin(\sqrt{\mu}t + g_1) \quad \text{and} \quad y = b \sin(\sqrt{\mu}t + g_2) \quad \text{where } a, b, g_1$$

and  $g_2$  are constants, so that  $\frac{dx}{dt} = a\sqrt{\mu} \cos(\sqrt{\mu}t + g_1)$

and  $\frac{dy}{dt} = b\sqrt{\mu} \cos(\sqrt{\mu}t + g_2).$

When  $t = 0$  then  $x = d$ ,  $y = 0$ ,  $\frac{dx}{dt} = V \cos \alpha$  and  $\frac{dy}{dt} = V \sin \alpha$ .

$\therefore d = a \sin g_1, \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$

$0 = b \sin g_2, \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$

$$\begin{aligned} V \cos \alpha &= a\sqrt{\mu} \cos g_1, & \dots & \dots & \dots & \dots & (3) \\ \text{and } V \sin \alpha &= b\sqrt{\mu} \cos g_2, & \dots & \dots & \dots & \dots & (4) \end{aligned}$$

$$\text{From (2), } g_2 = 0 \text{ and from (4), } b = \frac{V \sin \alpha}{\sqrt{\mu}}.$$

$$\text{From (1) and (3), } a^2 = d^2 + \frac{V^2 \cos^2 \alpha}{\mu} \text{ and } \sin g_1 = \frac{d}{a}.$$

Substituting  $\sin \sqrt{\mu} t = \frac{y}{b}$  in the equation

$$x = a \left[ \sin \sqrt{\mu} t \cos g_1 + \cos \sqrt{\mu} t \sin g_1 \right], \text{ we have,}$$

$$x = a \left[ \frac{y}{b} \cdot \sqrt{1 - \frac{d^2}{a^2}} + \sqrt{1 - \frac{y^2}{b^2}} \cdot \frac{d}{a} \right],$$

$$\text{i.e. } \left( x - \frac{y\sqrt{a^2 - d^2}}{b} \right)^2 = \frac{d^2(b^2 - y^2)}{b^2},$$

$$\text{i.e. } b^2 x^2 - 2bxy\sqrt{a^2 - d^2} + a^2 y^2 = d^2 b^2,$$

$$\text{where } a^2 = d^2 + \frac{V^2 \cos^2 \alpha}{\mu} \text{ and } b = \frac{V \sin \alpha}{\sqrt{\mu}}.$$

This is the equation of an ellipse having its centre at O.

In the particular case when  $\alpha = \frac{\pi}{2}$ ,  $a^2 = d^2$  and  $b = \frac{V}{\sqrt{\mu}}$  so that the equation of the path of the particle is

$$\frac{V^2}{\mu} x^2 + d^2 y^2 = \frac{d^2 V^2}{\mu},$$

$$\text{i.e. } \frac{x^2}{d^2} + \frac{y^2}{\frac{V^2}{\mu}} = 1.$$

The path is therefore an ellipse having its centre at O, its axes along OX and OY and the lengths of its semi-axes,  $d$  and  $\frac{V}{\sqrt{\mu}}$  respectively.

**19. Relative Velocity and Acceleration.**—At any instant the distance of one moving point from another, or the bearing of one with respect to the other, may be changing, or both may be changing. In any of these circumstances one point is said to have relative velocity with respect to the other. The rate of change of the relative velocity of one with respect to the other is the relative acceleration of the former point with respect to the latter. Given the velocities of the two points, the relative velocity can be determined both geometrically and analytically as follows.

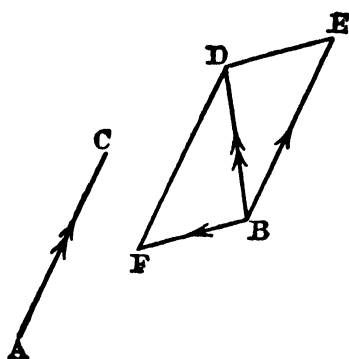


Fig. 19.

and thus the relative velocity of B with respect to A is determined completely by BF, whose magnitude and direction can be calculated by solving the triangle BDF.

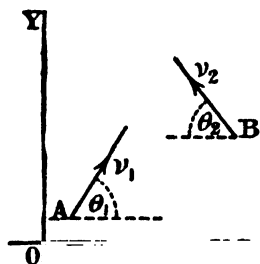


Fig. 20.

**Geometrical Method.**—A and B are the moving points and their velocities are represented in magnitude and direction by AC and BD respectively. BE is drawn equal and parallel to AC and the parallelogram BEDF is completed.

By the parallelogram of velocities, the velocity represented by BD is equivalent to the simultaneous velocities represented by BE and BF. The component BE neutralises the velocity of A

**Analytical Method.**—Let the velocities of A and B have magnitudes  $v_1$  and  $v_2$  and directions determined by  $\theta_1$  and  $\theta_2$ .

The component in direction OX, of the velocity of B relative to A is  $-v_2 \cos \theta_2 - v_1 \cos \theta_1$  and that in direction OY,  $v_2 \sin \theta_2 - v_1 \sin \theta_1$ .

$\therefore$  the relative velocity of B with respect to A has magnitude

$$[(-v_2 \cos \theta_2 - v_1 \cos \theta_1)^2 + (v_2 \sin \theta_2 - v_1 \sin \theta_1)^2]^{\frac{1}{2}},$$

i.e.  $(v_2^2 + v_1^2 + 2v_1v_2 \cos(\theta_1 + \theta_2))^{\frac{1}{2}}$

and direction, making with OX an angle

$$\tan^{-1} \frac{v_2 \sin \theta_2 - v_1 \sin \theta_1}{-v_2 \cos \theta_2 - v_1 \cos \theta_1}.$$

The question as to which of the two angles determined by the latter is the correct one, can be answered by consideration of the sign of  $(v_2 \sin \theta_2 - v_1 \sin \theta_1)$  for if  $+ve$ , the angle lies between  $0$  and  $180^\circ$  and if  $-ve$ , between  $180^\circ$  and  $360^\circ$ .

Both of the above methods apply equally well in determination of relative accelerations.

**Ex. III.** A ship A steams at 20 ml. per hr. in direction  $15^\circ$  E. of N. and a second ship B, at 12 ml. per hr.  $30^\circ$  W. of N. What is the relative velocity of B with respect to A? At 9.30 a.m. the ships



are 5 ml. apart and the bearing of B from A is  $25^\circ$  N. of E. If the ships steam uniformly on their course, find their shortest distance apart and the time when they are nearest one another.

### Method 1.

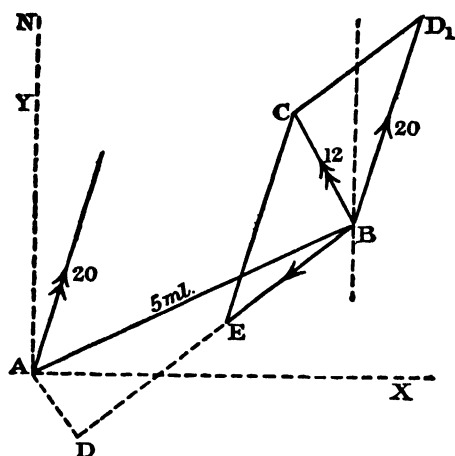


Fig. 21.

BC represents completely the velocity of B and  $BD_1$  represents the velocity of A in magnitude and direction. Then, completing the parallelogram  $BD_1CE$ , BE represents the velocity of B relative to A, in magnitude and direction.

$$BE^2 = 20^2 + 12^2 - 480 \cos 45^\circ,$$

$$\therefore BE = 14.3$$

$$\text{Also } \tan \hat{BEC} = \frac{12 \sin 45^\circ}{20 - 12 \cos 45^\circ}$$

$$\therefore \hat{BEC} = 36^\circ 23'$$

Thus the velocity of B relative to A is 14.3 ml. per hr. in direction  $38^\circ 37'$  S. of W.

AD is drawn at right angles to BE, meeting BE produced at D; thus AB representing 5 ml., AD represents the shortest distance apart.

$$AD = 5 \sin 13^\circ 37' = 1.1770.$$

Time when nearest one another is  $\frac{5 \cos 13^\circ 37'}{14.3}$  hr. after 9.30 a.m.,  
i.e. 20.4 min. after 9.30 a.m.

Thus, at 9.50 a.m., the ships are 1 ml. 312 yds. apart, which is the nearest distance.

### Method 2.

The velocity of B relative to A resolved along AX is

$$(-12 \sin 30^\circ - 20 \sin 15^\circ) \text{ ml. per hr.}$$

and resolved along AY,  $(12 \cos 30^\circ - 20 \cos 15^\circ)$  ml. per hr.

$\therefore$  the magnitude of the relative velocity of B with respect to A is  $[(11.176)^2 + (8.926)^2]^{\frac{1}{2}}$  ml. per hr.

$$\text{i.e. } 14.3 \text{ ml. per hr.}$$

The angle made with AX is

$$\tan^{-1} \frac{8.926}{11.176} = 38^\circ 37' \text{ or } (180^\circ + 38^\circ 37').$$

As  $-12 \sin 30^\circ - 20 \sin 15^\circ$  is  $-ve$ ,  $180^\circ + 38^\circ 37'$  is the required angle, and thus the direction of the velocity of B relative to A is  $38^\circ 37'$  S. of W.

The shortest distance apart and the time when the ships are in these positions could be found without determining the relative velocity, as follows.

Let  $s$  ml. be the distance apart after  $t$  hr. from 9.30 a.m.

$$\text{Then } s^2 = [5 \cos 25^\circ - t(12 \sin 30^\circ + 20 \sin 15^\circ)]^2 + [5 \sin 25^\circ + t(12 \cos 30^\circ - 20 \cos 15^\circ)]^2$$

$$= [4.5315 - 11.176t]^2 + [2.1130 - 8.926t]^2$$

$s$  is a minimum for the value of  $t$  given by  $\frac{d}{dt}(s^2) = 0$

$$\text{i.e. by } [11.176^2 + 8.926^2]t = 4.5315 \times 11.176 + 2.113 \times 8.926$$

$$\text{i.e. } 204.49t = 69.50$$

$$\text{i.e. } t = .3398$$

Subst. this value of  $t$  in the above expression for  $s^2$ , it is found that

$$s = 1.177.$$

Thus at 9.50 a.m., the ships are 1 ml. 312 yds. apart, which is the shortest distance.

Suppose O to be a fixed point and A,  $P_1$ ,  $P_2 \dots P_n$  to be moving points whose velocities in a given direction are  $v, v_1, v_2 \dots v_n$ .

Then  $v = (v - v_1) + (v_1 - v_2) + \dots + (v_{n-1} - v_n) + v_n$ .

Thus the velocity of A in the given direction is the sum of the velocities, in the same direction, of A relative to  $P_1$ , of  $P_1$  relative to  $P_2$ , and so on, and of the last relative to O.

$$\text{As } \frac{dv}{dt} = \frac{d}{dt}(v - v_1) + \frac{d}{dt}(v_1 - v_2) + \dots + \frac{d}{dt}(v_{n-1} - v_n) + \frac{dv_n}{dt},$$

the same principle applies for accelerations in a given direction.

**20. Angular Velocity and Acceleration.**—Suppose P is a point moving in a plane and that O is a fixed point in the plane and OX a fixed line also in the plane; then the rate of change of the angle XOP is the angular velocity of P about O and the rate of change of the angular velocity is the angular acceleration. If, after time  $t$ , the angle XOP be  $\theta$ , the angular velocity and acceleration of P about O will be  $\frac{d\theta}{dt}$

and  $\frac{d^2\theta}{dt^2}$  respectively. Also if  $\omega$  be the angular velocity, the angular acceleration will be  $\frac{d\omega}{dt}$  or  $\omega \frac{d\omega}{d\theta}$ .

Let P be the position of the point after time  $t$ , and Q that after time  $t + \delta t$ .

$$\text{Then, if } v \text{ be the velocity at P, } v = \lim_{\delta t \rightarrow 0} \frac{PQ}{\delta t}.$$

If  $\widehat{POQ} = \delta\theta$ ,  $OP = r$  and  $OQ = r + \delta r$ , then by expressing the area of the triangle  $OPQ$  in two ways, we have

$r(r + \delta r) \sin \delta\theta = PQ \cdot OA$ ,  
where  $OA$  is the perpendicular on  $PQ$ .

Dividing by  $\delta t$  and proceeding to the limit when  $\delta\theta \rightarrow 0$ ,

we have  $r^2 \frac{d\theta}{dt} = v\rho$

where  $\rho$  is the perpendicular to the tangent of the path at  $P$ .

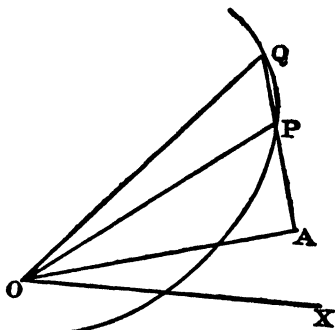


Fig. 22.

The angular acceleration is therefore  $\frac{d}{dt} \left( \frac{v\rho}{r^2} \right)$ .

Let the path of the point cut  $OX$  at  $O_1$ , then the **Areal Velocity** is the rate of change of the area  $O_1OP$  and is therefore

$$\lim_{\delta t \rightarrow 0} \frac{\text{Area } POQ}{\delta t} = \frac{1}{2} r^2 \cdot \frac{d\theta}{dt}.$$

The areal acceleration is  $\frac{d}{dt} \left( \frac{1}{2} r^2 \cdot \frac{d\theta}{dt} \right)$ .

**Ex. IV.**  $O$  is a fixed point on the circumference of a circle radius  $a$  and centre  $C$ , and a point  $P$  moves along the circumference with uniform velocity  $V$ . What is the angular velocity and the angular acceleration of  $P$  about  $O$  when  $\widehat{OCP} = \theta$ ?

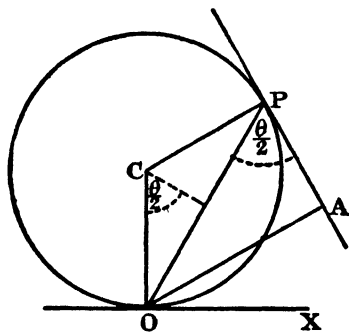


Fig 23

$OA$  is the perpendicular to the tangent at  $P$ .

Angular velocity of  $P$  about  $O$

$$\begin{aligned} &= \frac{V \cdot OA}{OP^2} \\ &= \frac{V \cdot 2a \sin^2 \frac{\theta}{2}}{4a^2 \sin^2 \frac{\theta}{2}} = \frac{V}{2a}. \end{aligned}$$

As this is constant, the angular acceleration is zero.

## 21. Radial and Transversal Velocities and Accelerations.—

Let  $r$ ,  $\theta$  be the polar co-ordinates of a moving point  $P$  after time  $t$ , and let  $Q$  whose co-ordinates are  $r + \delta r$ ,  $\theta + \delta\theta$  be the position after time  $t + \delta t$ .

Let  $u$  and  $v$  be the radial and transversal velocities at  $P$ , and  $u + \delta u$  and  $v + \delta v$ , the velocities at  $Q$ .

From Q, draw QA and QB perpendicular to OP and OQ respectively meeting OP at A and B.

$$\begin{aligned} \text{Then } u &= \lim_{\delta t \rightarrow 0} \frac{OA - OP}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \cos \delta \theta - r}{\delta t} \\ &= \frac{dr}{dt} \end{aligned}$$

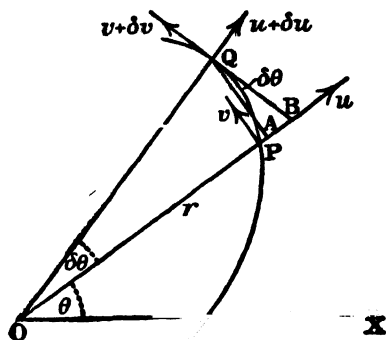


Fig. 24.

$$\text{and } v = \lim_{\delta t \rightarrow 0} \frac{AQ}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \sin \delta \theta}{\delta t} = r \frac{d\theta}{dt}.$$

Now the velocity at Q resolved along OP is

$$(u + \delta u) \cos \delta \theta - (v + \delta v) \sin \delta \theta.$$

And the velocity at Q resolved perp. to OP is

$$(u + \delta u) \sin \delta \theta + (v + \delta v) \cos \delta \theta.$$

$\therefore$  radial acceleration at P

$$\begin{aligned} &= \lim_{\delta t \rightarrow 0} \frac{(u + \delta u) \cos \delta \theta - (v + \delta v) \sin \delta \theta - u}{\delta t} \\ &= \frac{du}{dt} - v \frac{d\theta}{dt} \\ &= \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2. \end{aligned}$$

Also transverse acceleration at P

$$\begin{aligned} &= \lim_{\delta t \rightarrow 0} \frac{(u + \delta u) \sin \delta \theta + (v + \delta v) \cos \delta \theta - v}{\delta t} \\ &= u \frac{d\theta}{dt} + \frac{dv}{dt} \\ &= \frac{dr}{dt} \cdot \frac{d\theta}{dt} + \left( \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \right) \\ &= r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \text{ or } \frac{1}{r} \cdot \frac{d}{dt} \left( r^2 \frac{d\theta}{dt} \right) \end{aligned}$$

NOTE.  $\frac{dx}{dt}$  and  $\frac{d^2 x}{dt^2}$  are frequently denoted by  $\dot{x}$  and  $\ddot{x}$  and so with this notation, the radial and transverse velocities are  $\dot{r}$  and  $r\dot{\theta}$  and the accelerations  $\ddot{r} - r\dot{\theta}^2$  and  $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ .

If  $r$  is constant, the radial velocity is zero and the radial and transverse accelerations are  $-r\dot{\theta}^2$  and  $r\ddot{\theta}$ .

**Ex. V.** Two points are describing concentric circles of radii  $a$  and  $a'$  with angular velocities  $\omega$  and  $\omega'$  respectively. Prove that the angular velocity of the line joining them when its length is  $r$  is  $\{(r^2 + a^2 - a'^2)\omega + (r^2 - a^2 + a'^2)\omega'\}/2r^2$ .

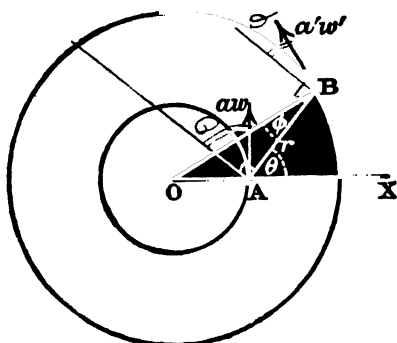


Fig. 25.

Let A, B be the positions of the points.

Let  $\widehat{BAX} = \theta$  and  $\widehat{OBA} = \phi$

Then, vel. of A at right angles to AB is  $a\omega \cos \theta$ , and vel. of B at right angles to AB is  $a'\omega' \cos \phi$ ;

$\therefore$  vel. of B in direction perp. to AB, relative to A is  $a'\omega' \cos \phi - a\omega \cos \theta$ .

$\therefore$  angular vel. of AB

$$= \frac{a'\omega' \cos \phi - a\omega \cos \theta}{r}.$$

Now  $a^2 = a'^2 + r^2 - 2a'r \cos \phi$  and  $a'^2 = a^2 + r^2 + 2ar \cos \theta$ .

$\therefore a' \cos \phi = (r^2 - a^2 + a'^2)/2r$  and  $a \cos \theta = -(r^2 + a^2 - a'^2)/2r$ .

$\therefore$  angular vel. of AB is

$$\{(r^2 + a^2 - a'^2)\omega + (r^2 - a^2 + a'^2)\omega'\}/2r^2.$$

**Ex. VI.** In the figure PAC represents in plan a window sash

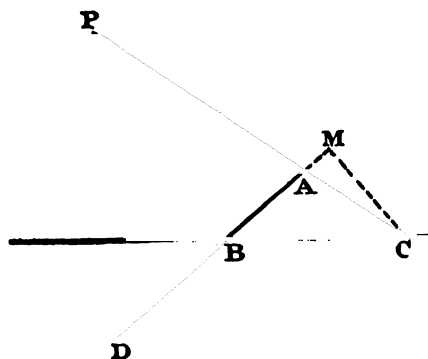


Fig. 26.

hinged at C, BC the window sill. A rod AD is hinged to the window sash at A, and can slide freely through a revolving joint fixed to the sill at B. If CM is the perpendicular from C on BA or BA produced, prove that when CA revolves about C with angular velocity  $\omega$ , the rod DA slides through B with velocity  $\omega$ . CM and at the same time rotates about B with

angular velocity  $\frac{\omega \cdot AM}{AB}$ .

Let  $v$  be the vel. with which DA slides through B and  $\omega'$  ( $\widehat{CBA}$  increasing) the ang. vel. of DA.

Let  $\widehat{BAP} = \theta$ , then vel. of A along BA =  $v$

and vel. of A perp. to BA =  $\omega' \cdot AB$

$\therefore$  vel. of A along CA =  $\omega' \cdot AB \sin \theta - v \cos \theta$

and vel. of A perp. to CA =  $\omega' \cdot AB \cos \theta + v \sin \theta$ .

But the vels. along and perp. to CA are 0 and  $\omega.CA$  respectively.

$$\therefore \omega'.AB \sin \theta - v \cos \theta = 0 \quad \dots \quad (1)$$

$$\text{and } \omega'.AB \cos \theta + v \sin \theta = \omega.CA \quad \dots \quad (2)$$

$$\therefore v = \omega.CA \sin \theta \text{ and } \omega'.AB = \omega.CA \cos \theta$$

$$\therefore v = \omega.CM \text{ and } \omega' = \frac{\omega.AM}{AB}.$$

**Ex. VII.** A and B are points on opposite banks of a river of breadth  $a$ , and AB is at right angles to the direction of the river. A boat leaves B and is rowed with constant velocity with the bow always directed towards A. If the velocity of the river be equal to this velocity, find the path of the boat.

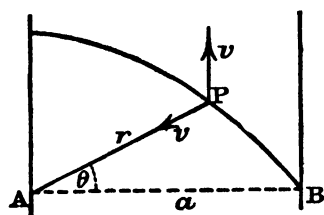


Fig. 27.

Let P be the position of the boat at any time.

Let  $AP = r$ ,  $\widehat{BAP} = \theta$ , and let  $v$  be the magnitude of each velocity.

$$\text{Then } \frac{dr}{dt} = -v + v \sin \theta,$$

$$\text{and } r \frac{d\theta}{dt} = v \cos \theta.$$

$$\therefore \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{-1 + \sin \theta}{\cos \theta}.$$

$$\therefore \int \frac{dr}{r} = \int [-\sec \theta + \tan \theta] d\theta.$$

$$\therefore \log r = -\log \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) - \log \cos \theta + \log C \text{ (a constant).}$$

When  $\theta = 0$ ,  $r = a$ , so that  $C = a$

$$\therefore r = \frac{a}{\tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \cdot \cos \theta}.$$

$$\text{Now } \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \cdot \cos \theta = \tan \left( \frac{\theta}{2} + \frac{\pi}{4} \right) \sin \left( \theta + \frac{\pi}{2} \right)$$

$$= 2 \sin^2 \left( \frac{\theta}{2} + \frac{\pi}{4} \right)$$

$$= 1 - \cos \left( \theta + \frac{\pi}{2} \right)$$

$$\therefore r = \frac{a}{1 - \cos \left( \theta + \frac{\pi}{2} \right)}.$$

This is the equation of a parabola having AB as semi-latus rectum.

**Ex. VIII.** A smooth horizontal tube OA of length  $a$  is movable about a vertical axis OB through the extremity O. A particle placed at the extremity A is suddenly projected towards O with velocity  $a\omega$  while at the same time the tube is made to revolve about OB with uniform angular velocity  $\omega$ . Show that the particle will have travelled half-way down the tube after a time  $\frac{1}{\omega} \log_e 2$ , and that it will not reach O in a finite time.

Let P(OP =  $r$ ) be the position after time  $t$  when the tube has rotated an angle  $\theta$ .

There are no forces acting along the tube

$$\therefore \frac{d^2 r}{dt^2} - r\omega^2 = 0.$$

Multiplying by  $2 \frac{dr}{dt}$  and integrating, we have

$$\left(\frac{dr}{dt}\right)^2 = r^2\omega^2 + C \text{ (a constant).}$$

When  $r = a$ ,  $\frac{dr}{dt} = -a\omega$  so that  $C = 0$  and thus  $\frac{dr}{dt} = -r\omega$ , the  $-ve$  sign being taken as the particle is moving towards O.

$$\therefore \int \frac{dr}{r} = -\omega \int dt$$

$$\therefore \log r = -\omega t + C_1, \text{ (a constant).}$$

When  $r = a$ ,  $t = 0$  so that  $C_1 = \log_e a$

$$\therefore t = \frac{1}{\omega} \log_e \frac{a}{r}.$$

When  $r = \frac{a}{2}$ ,  $t = \frac{1}{\omega} \log_e 2$  and when  $r \rightarrow 0$ ,  $t \rightarrow \infty$ .

## 22. Tangential and Normal Accelerations.—

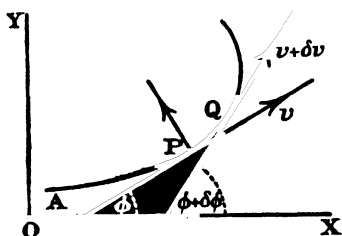


Fig. 28.

APQ is the path of a particle, A is a fixed point and P and Q are the positions after time  $t$ ,  $t + \delta t$  respectively. Arc AP =  $s$  and arc AQ =  $s + \delta s$ .

At P the particle moves in the direction of the tangent at P and its velocity

$$v = \lim_{\delta t \rightarrow 0} \frac{(s + \delta s) - s}{\delta t} = \frac{ds}{dt}.$$

Let  $v + \delta v$  be the velocity at Q, and let  $\phi$ ,  $\phi + \delta\phi$  be the angles the tangents at P, Q make with OX.

Tangential acceleration at P

$$= \lim_{\delta t \rightarrow 0} \frac{(v + \delta v) \cos \delta \phi - v}{\delta t} = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v \frac{dv}{ds}.$$

Normal acceleration at P

$$= \lim_{\delta t \rightarrow 0} \frac{(v + \delta v) \sin \delta \phi}{\delta t} = v \frac{d\phi}{dt} = v \frac{d\phi}{ds} \cdot \frac{ds}{dt}.$$

Now, if  $\rho$  be the radius of curvature of the curve at P,  
 $\rho = \frac{ds}{d\phi}.$

$\therefore$  normal acceleration at P =  $\frac{v^2}{\rho}$  in the direction towards the centre of curvature.

**23. Note on the Cycloid.**—A cycloid is the locus of a point on the circumference of a circle which rolls in a plane along a straight line.

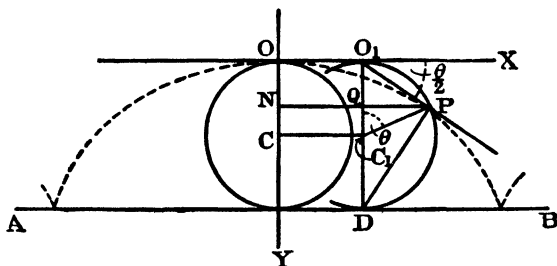


Fig. 29.

AB is the straight line on which the circle, radius  $a$ , rolls. O is the position of the point at its greatest distance from AB and P is the position when the circle has rotated an angle  $\theta$ . C and  $C_1$  are the positions of the centre of the circle in the two positions and OX and OY are the axes of co-ordinates.  $O_1C_1D$  is perpendicular to AB and  $PQN$  is parallel to AB.

As the circle has rolled without slipping, the distance moved by the centre is  $a\theta$ , i.e.  $CC_1 = a\theta$ . If  $x, y$  be the co-ordinates of P,

$$\begin{aligned} x &= PN = PQ + CC_1 = a(\theta + \sin \theta) \\ \text{and } y &= QO_1 = C_1O_1 - C_1Q = a(1 - \cos \theta) \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\sin \theta}{1 + \cos \theta} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \tan \frac{\theta}{2}.$$

$$\text{But } \frac{dy}{dx} = \tan \psi, \text{ where } \widehat{PO_1X} = \psi \quad \therefore \psi = \frac{\theta}{2}.$$



$$\therefore \text{ as } \widehat{PO_1X} = \frac{\theta}{2} \text{ and } \widehat{O_1PD} = \frac{\pi}{2},$$

the tangent to the cycloid at P passes through  $O_1$ , and the normal through D.

$$\begin{aligned} \text{Let arc } OP = s, \text{ then } \delta s^2 &= \delta x^2 + \delta y^2 \\ &= 2a^2(1 + \cos \theta) \cdot \delta \theta^2 = 4a^2 \cos^2 \frac{\theta}{2} \cdot \delta \theta^2. \end{aligned}$$

$$\therefore \int ds = 2a \int \cos \frac{\theta}{2} d\theta,$$

$$\therefore s = 4a \sin \frac{\theta}{2}, \text{ as } \theta = 0 \text{ when } s = 0.$$

$$\text{Again, as } y = 2a \sin^2 \frac{\theta}{2}, \sin \frac{\theta}{2} = \sqrt{\frac{y}{2a}} \text{ and } \therefore s = \sqrt{8ay}.$$

$$\text{Also, since } \psi = \frac{\theta}{2}$$

$$\therefore s = 4a \sin \psi.$$

$$\text{If } \rho \text{ be the radius of curvature, } \rho = \frac{ds}{d\psi} = 4a \cos \psi.$$

$$\text{Now } \widehat{PDO_1} = \frac{\theta}{2} = \psi \text{ and } \therefore PD = 2a \cos \psi; \text{ thus } \rho = 2.PD.$$

**Ex. IX.** A circle, radius  $a$ , starts from rest and rolls along a line, its centre moving with constant acceleration  $f$ . P is the point on the circumference initially in the highest position. Find the velocity and acceleration of P when the circle has rotated through an angle  $\frac{\pi}{2}$ .

O is the initial position of the point and P is the position after time  $t$  when the circle has rotated through an angle  $\theta$ .

$C_1$  the centre, has moved a distance  $a\theta$  in time  $t$ .

$$\therefore a\theta = \frac{1}{2}ft^2 \text{ and hence } a \frac{d\theta}{dt} = ft \text{ and } a \frac{d^2\theta}{dt^2} = f.$$

$$\text{Thus } t = \sqrt{\frac{2a\theta}{f}}, \quad \frac{d\theta}{dt} = \sqrt{\frac{2f\theta}{a}} \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \frac{f}{a}.$$

$$\text{Let } OP = s, \text{ then } s = 4a \sin \frac{\theta}{2}.$$

$$\text{Vel. of P} = \frac{ds}{dt} = 2a \cos \frac{\theta}{2} \cdot \frac{d\theta}{dt} = 2\sqrt{2af\theta} \cdot \cos \frac{\theta}{2} \text{ and its direc-}$$

tion makes angle  $\frac{\theta}{2}$  with OX.

$$\begin{aligned}\text{Tangential accel. of P} &= \frac{d^2s}{dt^2} = 2a \cos \frac{\theta}{2} \cdot \frac{d^2\theta}{dt^2} - a \sin \frac{\theta}{2} \left( \frac{d\theta}{dt} \right)^2 \\ &= 2f \cos \frac{\theta}{2} - 2f\theta \cdot \sin \frac{\theta}{2}.\end{aligned}$$

$$\text{Normal accel. of P} = \frac{\left( \frac{ds}{dt} \right)^2}{\rho} = \frac{8af\theta \cdot \cos^2 \frac{\theta}{2}}{4a \cos \frac{\theta}{2}} = 2f\theta \cdot \cos \frac{\theta}{2}.$$

When  $\theta = \frac{\pi}{2}$ , the vel. of P =  $\sqrt{2af\pi}$  in direction  $\frac{\pi}{4}$  with the given line.

$$\text{Also tang. accel. of P} = \sqrt{2}f - \frac{\sqrt{2}}{2}f\pi = \sqrt{2}f \cdot \left( 1 - \frac{\pi}{2} \right).$$

$$\text{And norm. accel. of P} = \frac{\sqrt{2} \cdot f\pi}{2}.$$

$$\begin{aligned}\therefore \text{resultant accel. of P} &= f \cdot \left[ 2 \left( 1 - \pi + \frac{\pi^2}{4} \right) + \frac{\pi^2}{2} \right]^{\frac{1}{2}} \\ &= f \cdot (2 - 2\pi + \pi^2)^{\frac{1}{2}}.\end{aligned}$$

Let  $\alpha$  be the acute angle the resultant acceleration makes with the given line.

$$\text{Then } \tan \left( \frac{\pi}{4} - \alpha \right) = \frac{\sqrt{2}f \left( \frac{\pi}{2} - 1 \right)}{\frac{\sqrt{2}f\pi}{2}} = \frac{\pi - 2}{\pi}.$$

$$\therefore \frac{1 - \tan \alpha}{1 + \tan \alpha} = \frac{\pi - 2}{\pi}.$$

$$\therefore \alpha = \tan^{-1} \frac{1}{\pi - 1}.$$

**Alternative Method.**

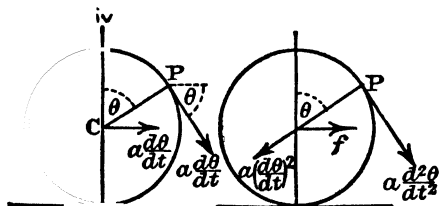


Fig. 30.

Also vel. of P perp. to given line

Vel. of P = vel. of P relative to C + vel. of C.

$\therefore$  vel. of P par. to given line

$$= a \frac{d\theta}{dt} \cos \theta + a \frac{d\theta}{dt}$$

$$= \sqrt{2af\theta} (1 + \cos \theta)$$

$$= \sqrt{2af\theta} \sin \theta.$$

$$\therefore \text{resultant vel. of P} = 2\sqrt{2af\theta} \cos \frac{\theta}{2}$$

$$\text{and direction with given line} = \tan^{-1} \frac{\sin \theta}{1 + \cos \theta} = \frac{\theta}{2}.$$

Accel. of P = accel. of P relative to C + accel. of C.

$\therefore$  accel. of P parl. to given line

$$= a \frac{d^2\theta}{dt^2} \cos \theta - a \left( \frac{d\theta}{dt} \right)^2 \sin \theta + f$$

$$= f(\cos \theta - 2\theta \sin \theta + 1).$$

Also accel. of P perp. to given line

$$= a \frac{d^2\theta}{dt^2} \sin \theta + a \left( \frac{d\theta}{dt} \right)^2 \cos \theta$$

$$= f(\sin \theta + 2\theta \cos \theta).$$

When  $\theta = \frac{\pi}{2}$ , these accelerations are  $(1 - \pi)f$  and  $f$  respectively.

$$\therefore \text{resultant accel of P} = f(2 - 2\pi + \pi^2)^{\frac{1}{2}}$$

$$\text{and direction with given line} = \tan^{-1} \frac{1}{1 - \pi} \text{ i.e. } \alpha = \tan^{-1} \frac{1}{\pi - 1}.$$

**Ex. X.** A point P describes a curve with constant velocity and its angular velocity about a given point O varies inversely as the distance from O; show that the curve is an equiangular spiral whose pole is O, and that the acceleration of the point varies inversely as OP.

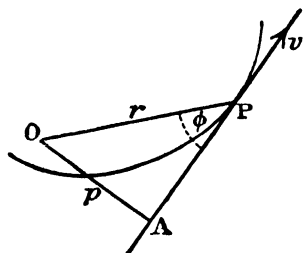


Fig. 31.

OA is the perp. to the tangent to the path at P.

Let  $OP = r$ ,  $OA = p$ ,  $\widehat{OPA} = \phi$  and  $v = \text{vel. of the point at P}$ .

$$\text{Then } \frac{v \sin \phi}{r} = \frac{\mu}{r} \text{ where } \mu \text{ is a constant.}$$

$\therefore \sin \phi = \frac{\mu}{v}$  and thus  $\phi$  is constant, therefore the curve is an equiangular spiral.

$$\text{As } p = r \sin \phi, \text{ the equation to the curve is } r = \frac{v}{\mu} p.$$

As  $\frac{dv}{dt} = 0$ , the acceleration is entirely along the normal.

$$\text{The radius of curvature } \rho \text{ is } r \frac{dr}{dp} = r \cdot \frac{v}{\mu}.$$

$$\therefore \text{ accel. of the point} = \frac{v^2}{\rho} = \frac{\mu v}{r} \text{ and thus varies inversely as } r.$$

## Problems III

1. A particle moves with acceleration  $2t$ ,  $3t$  ft. per sec. per sec. parallel to the axes  $OX$ ,  $OY$  respectively. Initially the particle is at the origin moving with velocity 4 ft. per sec. along the axis of  $x$ . Find the equation of the path and express the distance travelled in 2 seconds as a definite integral.

$$\left[ \left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right]$$

2. A particle is initially at the origin and moves with velocity  $V$  in a direction making angle  $\alpha$  with the axis of  $x$ . Its accelerations along the axes of  $x$  and  $y$  are zero and  $-2y$  respectively. Find the equation of the path and the wave-length.

3. At 10.0 a.m., from a ship A, a ship B can be seen in a direction  $15^\circ$  N. of E. and the distance apart is 6 miles. The velocity of A is 15 ml. per hr. N. and that of B, 18 ml. per hr.  $35^\circ$  W. of N. At what time will the ships be closest together and what is the distance then? What should the velocity of A have been in order that the ships should have collided?

4. A smooth horizontal tube  $OA$  of length  $a$  can rotate about  $O$ . A particle is placed in the tube at a distance  $b$  from  $O$  and the tube is then set rotating with constant angular velocity  $\omega$ . Find the velocity and direction with which the particle leaves the tube and also the time taken.

5. A circle radius  $a$  rolls on a fixed circle centre  $O$  and radius  $b$  with constant angular velocity  $\omega$  relative to the line of centres. Show that the angular velocity of the centre of the former about  $O$  is  $\frac{a}{b}\omega$ . A particle  $P$  on the rim of the moving circle is initially at distance  $(2a + b)$  from  $O$ . Find its resultant velocity and acceleration after time  $t$ , measuring the directions by the angles made with the initial position of the line  $OP$ .

6. An insect crawls at a constant rate  $u$  along the spoke of a cartwheel, of radius  $a$ , the cart moving with velocity  $V$ . Find the acceleration along and perpendicular to the spoke.

7. An arc lamp is 18 ft. from the ground. A man 6 ft. high starts from a point immediately under the lamp and walks at 5 ft. per second in a circle of radius 20 ft. Express the velocity and acceleration of the end of his shadow after  $t$  sec.

8. A point moves from the origin in such a way that its velocities in two fixed perpendicular directions after  $t$  seconds are  $a \cos t$  and  $a \sin t$ . Prove that its acceleration has a constant magnitude, and find the Cartesian equation of its path.

9. A circle of 2 in. radius rolls without slipping on a circle of radius 6 in.  $A$  is a point on the rolling circle. If the rolling circle makes a revolution in 4 seconds, find (i.) the angular speed of the radius of the fixed circle to the point of contact, (ii.) the speed of

the centre of the rolling circle, (iii.) the magnitude and direction of the velocity of A 1, 2, 3, 4 seconds after A is the point of contact. Plot out the path of A.

10. The position of a moving point after  $t$  seconds is given by the equations  $x = a \sin t$ ,  $y = x + b \cos t$ . Show that its acceleration may be represented by the straight line joining it to the origin, and that the sum of the squares of its acceleration and speed is constant.

11. A point A moves uniformly along the diameter of a circle whose centre is fixed and describes one diameter whilst the circle makes one revolution. Determine the path of A and the magnitude and direction of its velocity after 1, 2, 3, 4 sec. from the time A is on the circle, the angular velocity being 15 revolutions per minute and the diameter 15 cm.

12. Two rods AB and BC are hinged together at B, the end A being fixed, while the end C travels along a straight wire OP. The rods AB and BC make angles  $\theta$  and  $\phi$  respectively with AO, the perpendicular from A on OP, and CB produced meets AO in D. Prove that, if the rod AB turns round A with an angular velocity  $\omega$ , the angular velocity of the rod BC is (in magnitude)  $\frac{\omega \cdot BD}{BC}$ .

13. A crank AB length  $a$  revolves with uniform ang. vel.  $\omega$  about a fixed pt. A. The rod BC is of length  $b$  and C moves along a straight slot AC. Express the velocity and acceleration of C in terms of the angle BAC,  $\theta$ .

14. Two points are revolving in opposite directions in concentric circles of radii  $a$ ,  $b$ , with velocities  $u$ ,  $v$ . Prove that if the velocity of either particle relative to the other is along the straight line joining the particles the cosine of the angle which the particles subtend at the centre is  $(ua - vb)/(ub - va)$ .

15. Two ships are at opposite ends of a diameter of a circle 10 miles in radius. One sails at 2 miles per hour along the diameter, and the other at 4 miles per hour along the circumference. Find an equation determining the time at which they are first at a minimum distance apart, and solve this equation by the use of tables to within five minutes of time.

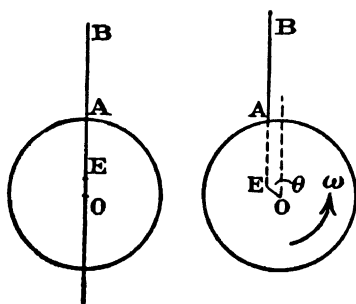


Fig. 32.

16. A particle Q moves along a straight line in such a way that its angular velocity about a fixed point A distant 5 cm. from the line is constantly equal to  $\frac{\pi}{36}$  radians per second. Find the velocity and acceleration of the particle 6 seconds from the instant it is in the position nearest to A.

17. A cam in the shape of a circle radius 8" rotates with uniform

velocity  $\omega$  about E which is  $2''$  vertically above O. A vertical rod in the same vertical line as E is then caused to move up and down. Express the velocity and acceleration of the rod in terms of  $\theta$ .

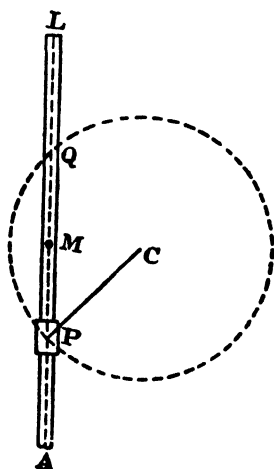
18. The line joining two points A, B is of constant length  $a$  and the velocities of A, B are in directions which make angles  $\alpha$  and  $\beta$  respectively with AB. Prove that the angular velocity of AB is  $\frac{u \sin (\alpha - \beta)}{a \cos \beta}$ , where  $u$  is the velocity of A.

19. A point moves in a circular path of radius  $a$  so that its angular velocity about a fixed point in the circumference of the circle is constant and equal to  $\omega$ ; show that the resultant acceleration of the point at every point of the path is of constant magnitude  $4a\omega^2$ .

20. A particle describes backwards and forwards an arc of a circle subtending an angle of  $2$  radians at the centre, so that its acceleration resolved along the tangent, is equal to  $\mu$  times its distance measured along the arc from the middle point; show that the least resultant acceleration is  $\frac{1}{2}\sqrt{3}\mu a$ , where  $a$  is the radius of the circle.

21. BC and OA are two rods hinged at O and  $OA = OB = OC$ . A is hinged at a fixed point and B moves along a fixed line XA starting from the point furthest from A. (i.) Find the path of C. (ii.) If B moves with uniform speed  $b$  cm. per sec., find the distance C moves in  $t$  sec. (iii.) Find expressions for the speed and acceleration of C after  $t$  sec. and also the angular velocity and acceleration of AO. When is the angular velocity of AO a maximum?

22. A rod CP turns with constant angular velocity  $\omega$  about C, and



by means of a slot at P guides a rod AL which is free to turn about A, the motions being entirely in one plane. Q is the point where the direction of the rod AL again meets the circle described by P, and M is the middle point of PQ. Find the angular velocity  $\omega'$  of CQ, and prove that if  $\Omega$  is the angular velocity of AL,  $\frac{\Omega^2}{\omega \cdot \omega'} = \frac{PM^2}{AP \cdot AQ}$ .

23. A point describes a cycloid with uniform speed. Show that the normal acceleration at any point varies inversely as the square root of the distance from the base of the cycloid.

24. A train is travelling at 60 miles an hour along a straight line of railway. Find its angular velocity about a point distant 1 mile from the railway two

Fig. 33.

minutes after the instant when the train is nearest to the point in question. Give the answer in degrees per minute.

Taking the more general case where  $v$  is the velocity of the train,  $a$  the distance of the point from the line,  $x$  the distance the train has travelled from the place where it was nearest the point, prove that the angular velocity of the train about the point in radians per unit time is  $va/(a^2 + x^2)$  and deduce the formula

$$\frac{d}{dx} \tan^{-1} \frac{x}{a} = \frac{a}{a^2 + x^2}.$$

25. Two rods  $O_1A_1$ ,  $O_2A_2$  are hinged at the fixed points  $O_1$ ,  $O_2$  respectively and a small ring fastened at  $A_1$  slides on the second rod. The rods are rotating with angular velocities  $\omega_1$ ,  $\omega_2$  respectively. A line  $A_1P$  is drawn at right angles to  $O_2A_2$  to meet  $O_1O_2$  at  $P$ ; prove that  $\omega_1.O_1P = \omega_2.O_2P$ .

26. A and B are two planets in conjunction and their paths are circles. A's period is 684 days and its distance from the centre is 1.52 times that of B, whose period is 365 days. Find the angular speed of A as viewed from B after any time " $t$ " days from the position of conjunction.

27. A particle moves in a plane under a constant acceleration  $\mu a$  parallel to  $Ox$ , and an acceleration  $-2\mu y$  parallel to  $Oy$ , where  $Ox$ ,  $Oy$  are rectangular axes. If the particle starts from rest

at the point  $(0, a)$  show that, when  $y$  is first equal to  $\frac{a}{2}$ ,  $x$  is equal to  $\frac{\pi^2 a}{36}$ .

28. If  $x = a(\cos \theta + \theta \sin \theta)$ ,  $y = a(\sin \theta - \theta \cos \theta)$ , and  $\theta$  increases at a uniform rate  $\omega$ , prove that the velocity of the point is  $a\theta\omega$ ; and find the inclination of the velocity to the axis of  $x$ .

29. A point moves in a plane so that its rectangular co-ordinates at time " $t$ " are  $(c \cos^3 nt, c \sin^3 nt)$ . Prove that the tangent to its path at the same time passes through the point  $(\frac{1}{3}c \cos nt, \frac{1}{3}c \sin nt)$ .

30. A particle moves with an acceleration which is always perpendicular towards, and equal to  $\mu$  divided by the cube of the distance from, a given straight line  $Ox$ . It is projected with velocity  $\frac{\sqrt{\mu}}{a}$  parallel to  $Ox$  from a point whose perpendicular distance from  $Ox$  is  $a$ . Show that its path is a portion of a circle.

31. A tube of length  $l$  is compelled to rotate in a horizontal plane with uniform angular velocity  $\omega$  about an end  $O$ , which is fixed. A particle inside the tube, the surface of which is smooth, is released from rest at a distance  $\frac{1}{2}l$  from  $O$ . Find the time which elapses before the particle leaves the tube, and find the angle between its direction of motion and the tube at the instant of leaving.

## CHAPTER IV

### MOTION OF A PARTICLE IN ONE PLANE II

#### 24. Motion of a Particle under the Action of Forces in One Plane.—

(a) Let the resolved parts of the forces parallel to the axes  $Ox$ ,  $Oy$ , be  $X$  and  $Y$  respectively and let the co-ordinates of the particle (of constant mass  $m$ ) at time  $t$  be  $x$  and  $y$ , then

$$X = m \frac{d^2x}{dt^2} \text{ and } Y = m \frac{d^2y}{dt^2}.$$

If  $X$  and  $Y$  be given, the velocity and path can be determined; conversely if the direction of the resultant force be given and also the equation of the path, the law of force can be determined.

(b) Let the resolved parts of the forces parallel and perpendicular to the radius vector be  $R$  and  $P$  respectively, then  $r$ ,  $\theta$  being the co-ordinates of the particle,

$$R = m \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \text{ and } P = m \left[ r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \right].$$

If  $R$  and  $P$  be given, the velocity and polar equation of the path can be determined.

(c) Let the resolved parts of the forces along the tangent and normal be  $T$  and  $N$  respectively, then the co-ordinates of the particle being  $s$  and  $\psi$  and  $v$ , the velocity,

$$T = m \frac{d^2s}{dt^2} \left( \text{or } m \frac{v dv}{ds} \right) \text{ and } N = m \frac{v^2}{\frac{ds}{d\psi}}.$$

NOTE.—The general solutions of the two equations in each of the above cases involve differential equations and are dealt with in Chapter XIV. Simple particular cases are considered in this chapter.

**25. Angular Momentum.**—The angular momentum of a particle of mass  $m$  about a point  $O$  is the product of its linear momentum and the perpendicular from  $O$  to the tangent of its path at the instant under consideration.

Let  $P$  be the position of the particle at time  $t$ ,  $OA$  the perpendicular to the tangent at  $P$ , and let  $\phi$  be the angle the tangent at  $P$  makes with  $Ox$ , then the angular momentum about  $O$

$$= mv \cdot OA.$$



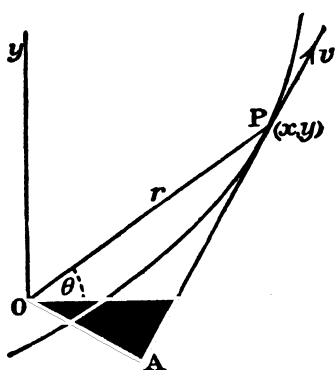


Fig. 34.

If  $X = 0$  and  $Y = 0$ , the angular momentum about  $O$  will be constant. Again, if the resultant force passes through  $O$ , as the moment about  $O$  will be zero, the angular momentum will be constant.

The velocity at right angles to  $OP$  is  $r \frac{d\theta}{dt}$  and therefore the angular momentum is  $mr^2 \frac{d\theta}{dt}$ . If the particle is under the action of a force whose line of action always passes through  $O$ ,  $mr^2 \frac{d\theta}{dt}$  and therefore  $\frac{1}{2} r^2 \frac{d\theta}{dt}$  will be constant. Thus the areal velocity is constant, that is, equal areas are swept over by the radius vector in equal times.

For a system of particles, we have

$$\Sigma X = \Sigma \left( m \frac{d^2 x}{dt^2} \right), \quad \Sigma Y = \Sigma \left( m \frac{d^2 y}{dt^2} \right),$$

$$\Sigma \left[ m \left( x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) \right] = \Sigma (xY - yX)$$

where  $\Sigma X$  and  $\Sigma Y$  are the algebraic sum of the extraneous forces acting on the system parallel to the axes of co-ordinates, for the internal forces being everywhere equal and opposite, their algebraic sum vanishes as also does the algebraic sum of their moments about  $O$ .

If  $\Sigma X = 0$  and  $\Sigma Y = 0$ , and  $\Sigma (xY - yX) = 0$ , then

$\Sigma \left( m \frac{dx}{dt} \right)$ ,  $\Sigma \left( m \frac{dy}{dt} \right)$  and  $\Sigma \left[ m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right]$  are constant.

Let  $\Sigma m = M$  and let the co-ordinates of the centre of inertia be  $\bar{x}$ ,  $\bar{y}$ ,

$$= mv (x \sin \phi - y \cos \phi)$$

$$= m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right)$$

$\therefore$  rate of change of angular momentum

$$= \frac{d}{dt} \left[ m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right]$$

$$= m \left( x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right)$$

$$= xY - yX$$

= moment of the forces about  $O$ .

then  $\Sigma(mx) = M\bar{x}$  and  $\Sigma(my) = M\bar{y}$   
 so that  $\Sigma\left(m\frac{dx}{dt}\right) = M\frac{d\bar{x}}{dt}$  and  $\Sigma\left(m\frac{dy}{dt}\right) = M\frac{d\bar{y}}{dt}$ .

Therefore  $\frac{d\bar{x}}{dt}$  and  $\frac{d\bar{y}}{dt}$  are constant, so that if no extraneous forces act on the system, the centre of inertia is at rest or moves with constant velocity in a straight line.

$\Sigma\left[m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right]$  is the algebraic sum of the angular momenta of the particles about O and therefore if  $\Sigma(xY - yX) = 0$ , the total angular momentum will be constant. Thus if no extraneous forces act on a system of particles the total angular momentum about any fixed point is constant. This is known as the **Principle of the Conservation of Angular Momentum**. Again if the extraneous forces be such that they can be always reduced to a single force acting through a fixed point, then the total angular momentum about this point is constant.

NOTE I. As  $\Sigma\left(m\frac{dx}{dt}\right) = M\frac{d\bar{x}}{dt}$   $\therefore \Sigma\left(m\frac{d^2x}{dt^2}\right) = M\frac{d^2\bar{x}}{dt^2}$  and thus  $\Sigma X = M\frac{d^2\bar{x}}{dt^2}$  and similarly  $\Sigma Y = M\frac{d^2\bar{y}}{dt^2}$ . Hence the centre of inertia of a system of particles moves as if the whole mass were collected at that point, and as if all the external forces were applied there parallel to their original directions. It is proved in Article 62 that the rate of change of angular momentum of a system of particles about the centre of inertia is the same as if the centre of inertia were fixed and the same forces acted on the system. If therefore, the resultant force always passes through the centre of inertia, the angular momentum of the system about the centre of inertia, whatever its motion, is constant.

Ex. 1. A smooth wedge, length  $l$ , angle  $\alpha$  and mass  $M$  rests on a smooth horizontal table. A particle of mass  $m$  is placed at the top and allowed to move. Find the distance moved by the wedge when the particle reaches the bottom. Also find the time taken.

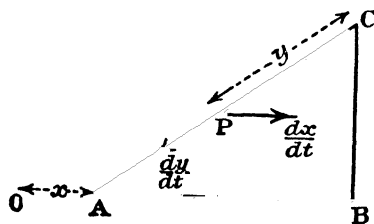


Fig. 35.

After time  $t$ , suppose the wedge to have moved a distance  $x$  and the particle a distance  $y$  relative to the wedge.

The horizontal external force acting on the system being zero,

$$\therefore (M + m)\frac{dx}{dt} - m\frac{dy}{dt}\cos\alpha = 0. \quad \dots \quad \dots \quad \dots \quad (1)$$

As the wedge is smooth, the only force acting on the particle down the plane is  $mg \sin \alpha$ .

$$\therefore m \left( \frac{d^2 y}{dt^2} - \frac{d^2 x}{dt^2} \cos \alpha \right) = mg \sin \alpha \quad \dots \quad (2)$$

From (1), integrating,  $(M + m)x = my \cos \alpha$ .  
(Constant vanishes as  $x = 0$  when  $y = 0$ .)

$$\therefore \text{when } y = l, x = \frac{ml \cos \alpha}{M + m}.$$

$$\text{Also from (1),} \quad (M + m) \frac{d^2 x}{dt^2} = m \cos \alpha \cdot \frac{d^2 y}{dt^2}.$$

$$\therefore \text{from (2),} \quad m \frac{d^2 y}{dt^2} \left( 1 - \frac{m \cos^2 \alpha}{M + m} \right) = mg \sin \alpha$$

$$\therefore \frac{d^2 y}{dt^2} = \frac{(M + m)g \sin \alpha}{M + m \sin^2 \alpha}.$$

$\therefore$  time taken for the particle to reach the bottom

$$= \left[ \frac{2l(M + m \sin^2 \alpha)}{(M + m)g \sin \alpha} \right]^{\frac{1}{2}}.$$

**Ex. II.** Particles whose masses are  $m$  and  $M$  respectively are connected by an inextensible string of length  $a$ . Initially the string is taut and the mass  $m$  is projected with velocity  $V$  at right angles to the string. What is the distance of each particle from the original positions when the string has made a complete rotation? Also find the tension of the string.

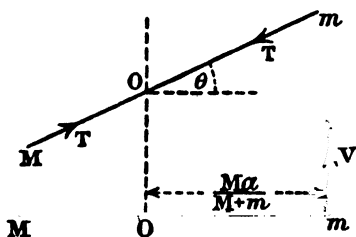


Fig. 36.

As no external forces act on the system the linear momentum is constant.

Initially the linear momentum is  $mV$  at right angles to the string.

Let  $O$  be the centre of inertia of the system; then  $O$  moves with

constant velocity  $\frac{mV}{M + m}$  at right angles to the string.

Again, as no external forces act, the angular momentum relative to  $O$  is constant and as the masses are at constant distances from  $O$ , the angular velocity of the string is constant.

$$\text{Initial angular velocity of string} = \frac{V - \frac{mV}{M + m}}{\frac{Ma}{M + m}} = \frac{V}{a}.$$

Let  $\theta$  be the angle through which the string rotates in time  $t$ ,

$$\text{then} \quad \frac{d\theta}{dt} = \frac{V}{a}, \quad \therefore \theta = \frac{V}{a}t.$$

When  $\theta = 2\pi$ ,  $t = \frac{2\pi a}{V}$  and the distance of the particles from their initial positions is  $\frac{2\pi am}{M+m}$ .

Let  $T$  be the tension of the string after time  $t$ , then as the acceleration of  $O$  is zero,

$$\begin{aligned} T &= m \cdot \frac{Ma}{M+m} \cdot \left(\frac{d\theta}{dt}\right)^2 \\ &= \frac{mMV^2}{a(M+m)}, \text{ which is constant.} \end{aligned}$$

**28. Work and Kinetic Energy.**—A particle of mass  $m$  is under the action of conservative forces, that is forces whose magnitude and direction depend on the position of the particle and which are independent of the time and velocity.

Let the algebraic sum of the forces resolved parallel to the axes be denoted by  $f_1(x)$  and  $f_2(y)$  respectively and suppose that the particle moves along a path APQB.

Let the co-ordinates of  $P$  be  $(x)$ ,  $(y)$  and of  $Q$ ,  $(x+\delta x)$ ,  $(y+\delta y)$ , then the work done by the forces in moving the particle from  $P$  to  $Q$  is  $f_1(x)\delta x + f_2(y)\delta y$ .

If the co-ordinates of  $A$  and  $B$  be  $x_1, y_1$  and  $x_2, y_2$  respectively, the work done by the forces in moving the particle from  $A$  to  $B$  is

$$\int_{x_1}^{x_2} f_1(x) dx + \int_{y_1}^{y_2} f_2(y) dy$$

which is constant whatever the path moved over by the particle from  $A$  to  $B$ .

Let  $v_1, v_2$  be the velocities of the particle at  $A, B$  respectively.

Then the gain of Kinetic Energy

$$\begin{aligned} &= \frac{1}{2} m (v_2^2 - v_1^2) \\ &= \frac{1}{2} m \left[ \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2 - \left(\frac{dx_1}{dt}\right)^2 - \left(\frac{dy_1}{dt}\right)^2 \right] \\ &= \frac{1}{2} m \left[ \left(\frac{dx_2}{dt}\right)^2 - \left(\frac{dx_1}{dt}\right)^2 \right] + \frac{1}{2} m \left[ \left(\frac{dy_2}{dt}\right)^2 - \left(\frac{dy_1}{dt}\right)^2 \right] \\ &= \int_{x_1}^{x_2} f_1(x) dx + \int_{y_1}^{y_2} f_2(y) dy. \quad [\text{Cf. Art. 12.}] \\ &= \text{work done by the forces.} \end{aligned}$$

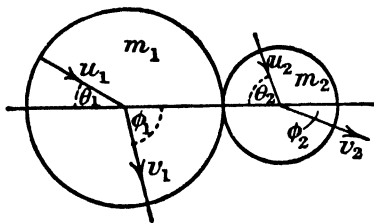
**27. Impact.**

Fig. 37.

Two smooth spheres, masses  $m_1$  and  $m_2$  respectively, impinge on one another. Just before impact the velocities are  $u_1, u_2$  in directions making angles  $\theta_1, \theta_2$  with the line joining their centres.

The co-efficient of restitution is  $e$ .

Let  $v_1, v_2$  be the velocities just after impact in directions inclined at angles  $\phi_1, \phi_2$  with the line joining the centres.

As there is no force perpendicular to the line joining the centres the velocities in that direction are unaltered.

$$\text{Hence } u_1 \sin \theta_1 = v_1 \sin \phi_1 \quad \dots \quad (1)$$

$$\text{and } u_2 \sin \theta_2 = v_2 \sin \phi_2 \quad \dots \quad (2)$$

Again the momentum of the system along the line joining the centre is constant as the forces at the point of contact are equal and opposite.

Hence

$$m_1 u_1 \cos \theta_1 + m_2 u_2 \cos \theta_2 = m_1 v_1 \cos \phi_1 + m_2 v_2 \cos \phi_2 \quad \dots (3)$$

By Newton's experimental law, we have

$$e(u_1 \cos \theta_1 - u_2 \cos \theta_2) = v_2 \cos \phi_2 - v_1 \cos \phi_1 \quad \dots (4)$$

From these four equations  $v_1, v_2, \phi_1$  and  $\phi_2$  can be determined.

It is left as an exercise to the student to prove that

(1) Impulse of the blow

$$= \frac{m_1 m_2}{m_1 + m_2} (1 + e)(u_1 \cos \theta_1 - u_2 \cos \theta_2).$$

(2) Loss of Kinetic Energy

$$= \frac{1 - e^2}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} (u_1 \cos \theta_1 - u_2 \cos \theta_2)^2.$$

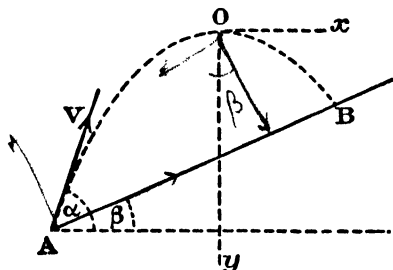
**28. Motion of a Projectile under Gravity (Air-resistance neglected).—**

Fig. 38.

A particle (mass  $m$ ) is projected from A with velocity  $V$  in direction inclined at angle  $\alpha$  to the horizontal.

Let O, the highest point of the path, be taken as origin. As no force acts horizontally, the horizontal velocity is constant and equal to  $V \cos \alpha$ .

Let  $P(x, y)$  be the position of the particle  $t$  sec. after leaving  $O$ ,  
 then  $x = V \cos \alpha \cdot t$   
 and  $y = \frac{1}{2} g t^2$ ;

$$\therefore x^2 = \frac{2V^2 \cos^2 \alpha}{g} y,$$

which is the equation of a parabola whose axis is vertical and whose latus rectum is

$$\frac{2V^2 \cos^2 \alpha}{g}.$$

Let  $R$  be the range up an incline making angle  $\beta$  with the horizontal then, resolving at right angles to the plane,

$$0 = V \sin (\alpha - \beta) \cdot t' - \frac{1}{2} g \cos \beta \cdot t'^2 \quad \dots \quad (1)$$

where  $t'$  is the time of flight.

Also, resolving horizontally,

$$\begin{aligned} R \cos \beta &= V \cos \alpha \cdot t' \quad \dots \quad \dots \quad \dots \quad (2) \\ \therefore R &= \frac{2V^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta} \\ &= \frac{V^2}{g \cos^2 \beta} [\sin (2\alpha - \beta) - \sin \beta]. \end{aligned}$$

The range is a maximum when  $2\alpha - \beta = \frac{\pi}{2}$ , i.e., when  $\alpha = \frac{\pi}{4} + \frac{\beta}{2}$ , so that a maximum range is obtained when the particle is projected in a direction bisecting the angle between the plane and the vertical.

$$\text{Also maximum range} = \frac{V^2}{g \cos^2 \beta} [1 - \sin \beta] = \frac{V^2}{g(1 + \sin \beta)}.$$

By putting  $-\beta$  for  $\beta$ , the maximum range down the plane is

$$\frac{V^2}{g(1 - \sin \beta)}.$$

**Ex. III.** A small ball is projected into the air; show that it appears to an observer standing at the point of projection to fall past a given vertical plane with constant velocity.

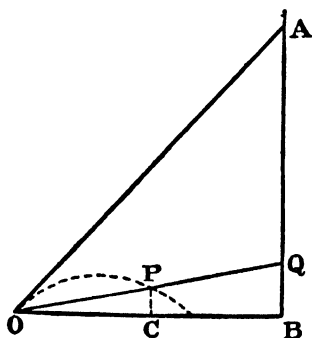


Fig. 39.

$O$  is the point of projection,  $AB$  the vertical plane and  $P$ , the position of the ball after time  $t$ .  $A$  is the point where the initial direction meets the plane,  $Q$  is the point where  $OP$  meets the plane and  $PC$  is the vertical through  $P$ .

Let  $OB = d$ ,  $\angle AOB = \alpha$ ,  $V = \text{vel. of projection}$  and  $AQ = x$ .

Then  $\frac{d \tan \alpha - x}{d} = \frac{V \sin \alpha \cdot t - \frac{1}{2}gt^2}{V \cos \alpha \cdot t},$

$\therefore x = d \tan \alpha - \frac{d}{V \cos \alpha} \left[ V \sin \alpha - \frac{1}{2}gt \right]$

$\therefore \frac{dx}{dt} = \frac{dg}{2V \cos \alpha},$  which is constant.

Thus the velocity of Q is constant.

**Ex. IV.** If any number of particles are projected simultaneously from the same point with equal velocities in the same vertical plane, prove that at any instant the particles all lie on a circle; also that their relative directions as seen from one another remain unchanged.

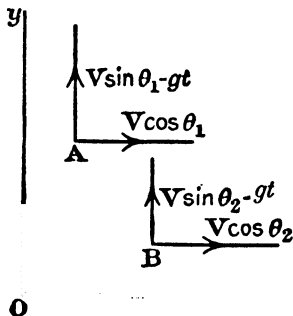
Let  $V$  be the velocity of projection and let  $x, y$  be the co-ordinates (referred to the point of projection as origin and axes horizontal and vertical) after time  $t$  of the particle whose direction of projection makes angle  $\theta$  with the horizontal.

Then  $x = V \cos \theta \cdot t \quad \dots \quad \dots \quad \dots \quad (1)$

and  $y = V \sin \theta \cdot t - \frac{1}{2}gt^2 \quad \dots \quad \dots \quad \dots \quad (2)$

Eliminating  $\theta$  from equations (1) and (2) we have that at the instant, time  $t$  from the moment the particles are projected, all the particles lie on the circle whose equation is

$$x^2 + (y + \frac{1}{2}gt^2)^2 = V^2t^2.$$



Let A and B be the positions after time  $t$  of particles whose directions of projection make angles  $\theta_1, \theta_2$  respectively with the horizontal.

Then the vertical velocities are  $(V \sin \theta_1 - gt)$  and  $(V \sin \theta_2 - gt)$  and the horizontal velocities,  $V \cos \theta_1$  and  $V \cos \theta_2$ .

$\therefore$  the direction of motion of B relative to A is

$$\tan^{-1} \left[ \frac{(V \sin \theta_2 - gt) - (V \sin \theta_1 - gt)}{V \cos \theta_2 - V \cos \theta_1} \right] \\ = \tan^{-1} \left[ \frac{\sin \theta_2 - \sin \theta_1}{\cos \theta_2 - \cos \theta_1} \right]$$

which is independent of  $t$ , thus showing that the relative direction of any one particle as seen from any other remains unchanged.

**NOTE 2.** The above angle simplifies to  $-\left(\frac{\pi}{2} - \frac{\theta_1 + \theta_2}{2}\right)$  and the magnitude of the relative velocity to  $2V \sin \frac{\theta_1 + \theta_2}{2}$ . Thus the velocity of any particle relative to any other is uniform in magnitude and direction.

**Ex. V.** Prove that the envelope of the paths of particles projected in vacuo from the same point with the same velocity in the same vertical plane, is a parabola with the point of projection as focus.

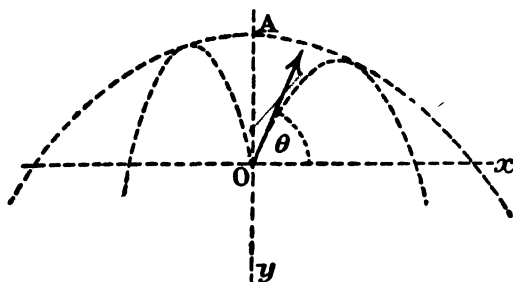


Fig. 41.

O is the point of projection, V is the initial velocity.

Let  $x, y$  (axes as shown) be the co-ordinates after time  $t$  of the particle projected in direction making angle  $\theta$  with the horizontal.

$$\text{Then } x = V \cos \theta \cdot t \quad \dots \quad (1)$$

$$\text{and } y = -V \sin \theta \cdot t + \frac{1}{2} g t^2 \quad \dots \quad (2)$$

Eliminating  $t$ , we find the equation of the path to be

$$y = -x \tan \theta + \frac{g x^2}{2 V^2} (1 + \tan^2 \theta).$$

Let  $\tan \theta = m$ , then the equation of the path is

$$y = -mx + \frac{g x^2}{2 V^2} (1 + m^2). \quad \dots \quad (3)$$

Differentiating partially with respect to  $m$ , we have

$$0 = -x + \frac{g x^2}{V^2} m. \quad \dots \quad (4)$$

Eliminating  $m$  from (3) and (4), the equation of the envelope of the family of parabolas given by (3) is

$$x^2 = \frac{2 V^2}{g} \left( y + \frac{V^2}{2g} \right),$$

which is a parabola whose axis is vertical and whose latus rectum is  $\frac{2 V^2}{g}$ . As  $y = \frac{1}{2} \cdot \frac{2 V^2}{g}$  when  $x = 0$ , therefore OA is  $\frac{1}{2}$  of the latus rectum, i.e., O is the focus of this parabola which is touched by the paths of all the particles.

## 29. Motion of a Particle on a Smooth Curve under Gravity.

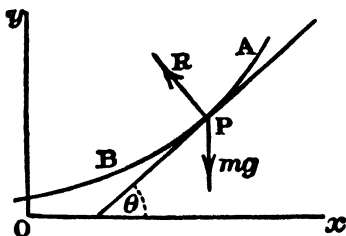


Fig. 42.

Let P be the position of the particle (mass  $m$ ), then the forces acting on it are  $mg$  vertically downwards and  $R$  at right angles to the tangent at P.

With the usual notation we have, by resolving along the tangent,



$$mv \frac{dv}{ds} = -mg \sin \theta \quad \dots \quad (1)$$

and by resolving along the normal,

$$m \frac{v^2}{\rho} = R - mg \cos \theta. \quad \dots \quad (2)$$

If the velocities at A and B be  $v_1$  and  $v_2$  respectively and A be a distance  $h$  vertically above B, then as gravity is a conservative force and R does no work during the motion we have

$$\frac{1}{2}m(v_2^2 - v_1^2) = mgh. \quad \dots \quad (3)$$

This equation can be derived from (1) by putting  $\sin \theta = \frac{dy}{ds}$ , integrating and substituting initial conditions.

There are thus two independent equations and in practice, problems are more readily solved by using the equation of energy and the equation obtained by resolving normally.

*Motion on a Smooth Circle.*

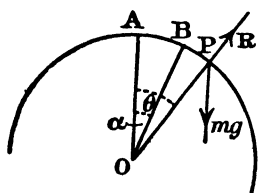


Fig. 43.

O is the centre of the circle (radius  $a$ ), A is the highest point, B ( $\widehat{AOB} = \alpha$ ) is the initial position where the velocity is  $V$  and P ( $\widehat{AOP} = \theta$ ) is the position after time  $t$ .

With the usual notation we have

$$\frac{1}{2}m[v^2 - V^2] = amg[\cos \alpha - \cos \theta] \quad \dots \quad (1)$$

$$\text{and} \quad m \frac{v^2}{a} = mg \cos \theta - R; \dots \quad (2)$$

$$\begin{aligned} \therefore R &= mg \cos \theta - 2mg(\cos \alpha - \cos \theta) - \frac{mV^2}{a} \\ &= mg(3 \cos \theta - 2 \cos \alpha) - \frac{mV^2}{a}. \end{aligned}$$

At the instant when the particle leaves the plane,  $R = 0$  and thus the equation  $V^2 = ag(3 \cos \theta - 2 \cos \alpha)$  determines the value of  $\theta$  at the point where the particle leaves the plane.

If  $V = 0$ , eqn. (1) becomes  $a^2 \left( \frac{d\theta}{dt} \right)^2 = 2 ag(\cos \alpha - \cos \theta)$ .

Therefore the time taken in moving from B to  $B_1$  ( $\widehat{AOB}_1 = \alpha_1$ ,

is given by

$$\sqrt{\frac{a}{2g}} \int_{\alpha_1}^{\alpha} \frac{d\theta}{\sqrt{\cos \alpha - \cos \theta}}.$$

*Motion on a Smooth Cycloid, Axis Vertical and Vertex Downwards.*

Let A be the vertex and P the position of the particle (mass  $m$ ) after time  $t$  from A. Let arc AP =  $s$  and let the inclination of the tangent at P with the horizontal be  $\psi$ . Then,  $a$  being the radius of the generating circle,  $s = 4a \sin \psi$ .

$$\text{Resolving tangentially, } m \frac{d^2 s}{dt^2} = -mg \sin \psi$$

$$\therefore \frac{d^2 s}{dt^2} = -\frac{g}{4a} s.$$

Hence the motion is Simple Harmonic, the periodic time being  $4\pi \sqrt{\frac{a}{g}}$ . Thus if the particle be placed at any point on the curve and allowed to move from rest under gravity, it reaches A in time  $\pi \sqrt{\frac{a}{g}}$ .

Suppose the particle to move from rest at the cusp, then if  $v$  be its velocity and R the reaction at P,

$$m \frac{v dv}{ds} = -m \frac{g}{4a} s \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\text{and} \quad R - mg \cos \psi = m \frac{v^2}{\rho} \quad \dots \quad \dots \quad \dots \quad (2)$$

Integrating eqn. (1) and putting  $v = 0$  when  $s = 4a$ , we have

$$v^2 = -\frac{g}{4a} s^2 + 4ag.$$

Substituting in eqn. (2) and putting  $\rho = 4a \cos \psi$ ,

$$R = 2mg \cos \psi.$$

### 30. The Cycloidal Pendulum.

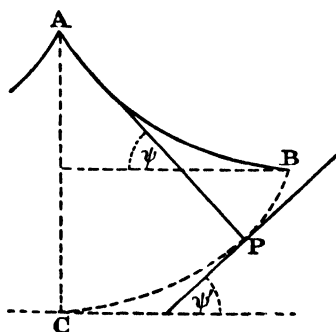


Fig. 44.

A is the cusp of a cycloid (axis vertical) whose equation is  $s = 4a \sin \psi$ . A particle (mass  $m$ ) is connected to A by a light string of length  $4a$  and allowed to oscillate.

Let CB be the path of the particle and P the position after time  $t$ .

The radius of curvature of the path at P is given by

$$\rho' = 4a - (4a - 4a \sin \psi) = 4a \sin \psi = 4a \cos \psi',$$

where  $\psi'$  is the inclination of the tangent to the path at P to the horizontal.

Let  $CP = s'$ , then

$$\frac{ds'}{d\psi'} = \rho' = 4a \cos \psi'$$

$$\therefore s' = 4a \sin \psi'.$$

Hence CPB is a cycloid, the radius of whose generating circle equals that of the cycloid AB.

Resolving at rt. angles to the string, we have

$$m \frac{d^2 s'}{dt^2} = -mg \sin \psi';$$

$$\therefore \frac{d^2 s'}{dt^2} = -\frac{g}{4a} s'.$$

Hence the time of oscillation is  $2\pi \sqrt{\frac{4a}{g}}$ , i.e.  $4\pi \sqrt{\frac{a}{g}}$ , which

is independent of the amount of swing and thus the Cycloidal Pendulum is Isochronous.

From the previous article, the tension of the string is  $2mg \cos \psi'$ .

**Ex. VI.** A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and the vertex downwards. Prove that the acceleration of the particle is of constant magnitude.

Let the equation of the cycloid be

$$s = 4a \sin \psi.$$

Then with the usual notation, resolving along the tangent we have

$$v \frac{dv}{ds} = -g \sin \psi = -\frac{g}{4a} s.$$

Integrating and putting  $v = 0$  when  $s = 4a$ ,

$$v^2 = 4ag - \frac{g}{4a} s^2 = \frac{g}{4a} (16a^2 - s^2).$$

Now, normal accel.  $= \frac{v^2}{\rho}$  and  $\rho = 4a \cos \psi = (16a^2 - s^2)^{\frac{1}{2}}$ .

$$\begin{aligned} \text{Resultant accel.} &= \left[ \left( v \frac{dv}{ds} \right)^2 + \frac{v^4}{\rho^2} \right]^{\frac{1}{2}} \\ &= \left[ \frac{g^2 s^2}{16a^2} + \frac{g^2 (16a^2 - s^2)}{16a^2} \right]^{\frac{1}{2}} = g. \end{aligned}$$

**NOTE.**—If  $\phi$  be the angle the direction of the resultant accel. makes with the normal, then  $\tan \phi = \frac{g s}{4a \cdot \frac{g \sqrt{16a^2 - s^2}}{s}} = \frac{s}{\sqrt{16a^2 - s^2}} = \tan \psi$ .  $\therefore \phi = \psi$ .

**Ex. VII.** A smooth cup is set whirling with angular speed  $\omega$  about its axis which is vertical. If a particle will remain in equilibrium on any part of the surface, find the equation to the curve of section.

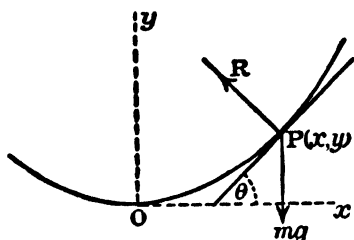


Fig. 45.

P is any position of the particle; let its co-ordinates referred to axes as shown, be  $x, y$ .

Let  $R$  be the reaction at  $P$ ,

$$\text{then resolving vertically, } R \cos \theta - mg = 0 \quad \dots \quad (1)$$

$$\text{and resolving horizontally, } R \sin \theta = m x \omega^2 \quad \dots \quad (2)$$

$$\therefore \tan \theta = \frac{\omega^2 x}{g}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{\omega^2 x}{g}$$

$$\therefore y = \frac{\omega^2}{2g} x^2 \text{ is the equation of the curve of section.}$$

**Ex. VIII.** A particle can move in a smooth circular tube which revolves about a fixed vertical tangent with uniform angular velocity  $\omega$ . Find the position of relative equilibrium of the particle and show that the time of a small oscillation about that position is  $\frac{2\pi}{\omega} \left\{ \frac{\sin \alpha}{1 + \sin^2 \alpha} \right\}^{\frac{1}{2}}$  where  $\alpha$  is the angle of inclination to the vertical of the radius to the particle when in relative equilibrium.

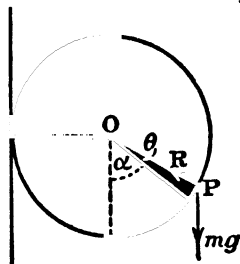


Fig. 46.

Let  $P$  be the position of the particle (mass  $m$ ) and let  $R$  be the reaction. Let  $\alpha + \theta$  be the angle the radius  $OP$  makes with the vertical.

When  $\theta = 0$ , we have

$$R \cos \alpha - mg = 0 \quad \dots \quad (1)$$

$$\text{and } R \sin \alpha = m a (1 + \sin \alpha) \omega^2, \quad \dots \quad (2)$$

where  $a$  is the radius of the tube.

$$\therefore \tan \alpha = \frac{a(1 + \sin \alpha) \omega^2}{g}, \quad \dots \quad (3)$$

which determines the position of relative equilibrium.

Resolving at right angles to  $OP$ , we have,

$$mg \sin (\alpha + \theta) = -ma \frac{d^2 \theta}{dt^2} + ma(1 + \sin \alpha + \theta) \omega^2 \cos (\alpha + \theta) \dots (4)$$

$$\therefore \frac{ad^2 \theta}{dt^2} = - \left[ g \sin (\alpha + \theta) - a \omega^2 \cos (\alpha + \theta) (1 + \sin \alpha + \theta) \right]$$

$$= - \left[ \frac{a \omega^2 (1 + \sin \alpha) \sin (\alpha + \theta)}{\tan \alpha} - a \omega^2 \cos (\alpha + \theta) (1 + \sin \alpha + \theta) \right], \text{ from (3)}$$

$$= -\frac{a\omega^2}{\sin \alpha} \left[ \cos \alpha (1 + \sin \alpha) \sin (\alpha + \theta) - \sin \alpha \cdot \cos (\alpha + \theta) (1 + \sin \alpha + \theta) \right].$$

Now  $\lim_{\theta \rightarrow 0} \cos \theta = 1$  and  $\lim_{\theta \rightarrow 0} \sin \theta = \theta$ ,

therefore when  $\theta$  is small so that terms in  $\theta^2$  may be neglected,

$$\begin{aligned} a \frac{d^2 \theta}{dt^2} = & -\frac{a\omega^2}{\sin \alpha} \left[ \cos \alpha (1 + \sin \alpha) (\sin \alpha + \theta \cos \alpha) \right. \\ & \left. - \sin \alpha (\cos \alpha - \theta \sin \alpha) (1 + \sin \alpha + \theta \cos \alpha) \right] \end{aligned}$$

so that 
$$\frac{d^2 \theta}{dt^2} = -\frac{\omega^2 (1 + \sin^3 \alpha)}{\sin \alpha} \cdot \theta.$$

Hence for small vibrations, the motion relative to the tube is simple harmonic and the periodic time is  $\frac{2\pi}{\omega} \left[ \frac{\sin \alpha}{1 + \sin^3 \alpha} \right]^{\frac{1}{2}}.$

**Ex. IX.** Find the law of force parallel to the axis of  $y$  under which a particle describes the curve  $xy = a^2$ .

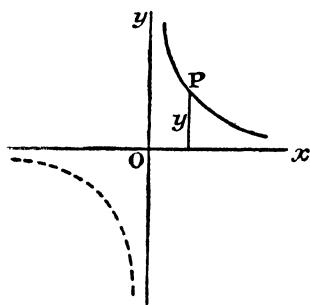


Fig. 47.

$\therefore Y = \frac{2u^2 m}{a^4} y^3$ , i.e., the force varies as the cube of the ordinate.

**Ex. X.** A heavy particle P slides on a smooth curve of any form in a vertical plane. The centre of curvature at P is Q and

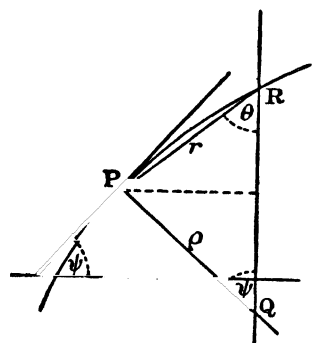


Fig. 48.

Let  $Y$  be the force and  $m$  the mass of the particle,

then  $m \frac{d^2 x}{dt^2} = 0$  and  $m \frac{d^2 y}{dt^2} = Y$ .

$\therefore \frac{dx}{dt} = u$ , a constant.

As  $y = \frac{a^2}{x}$

$$\therefore \frac{dy}{dt} = -\frac{a^2}{x^2} \cdot \frac{dx}{dt} = -\frac{a^2 u}{x^2}$$

$$\therefore \frac{d^2 y}{dt^2} = \frac{2a^2 u}{x^3} \cdot \frac{dx}{dt} = \frac{2a^2 u^2}{x^3} = \frac{2u^2}{a^4} \cdot y^3$$

R is on the same vertical as Q and at the level of zero velocity. Shew that the acceleration makes with the normal an angle

$$\tan^{-1} \left( \frac{1}{2} \tan \widehat{PRQ} \right).$$

Let  $PR = r$ ,  $PQ = \rho$ ,  $\widehat{PQR} = \psi$  and the inclination of  $PR$  to the vertical,  $\theta$ .

R being the initial position, we have

$$\frac{1}{2} m \left( \frac{ds}{dt} \right)^2 = mg r \cos \theta.$$

Also  $m \frac{d^2 s}{dt^2} = mg \sin \psi,$

so that the angle which the resultant acceleration makes with PQ is

$$\begin{aligned}\tan^{-1} \left( \frac{\frac{d^2s}{dt^2}}{\left(\frac{ds}{dt}\right)^2/\rho} \right) &= \tan^{-1} \left( \frac{\rho g \sin \psi}{2g r \cos \theta} \right) \\ &= \tan^{-1} \left( \frac{r \sin \theta}{2r \cos \theta} \right) = \tan^{-1} \left( \frac{1}{2} \tan \widehat{PRQ} \right).\end{aligned}$$

### Problems IV

1. Prove that the least velocity with which a projectile can be thrown so as to reach a point  $y$  feet high and distant  $x$  feet horizontally from the thrower is given by the formula  $v^2 = g[y + \sqrt{(x^2 + y^2)}]$  and the corresponding angle of elevation ( $\alpha$ ) of the direction of

throw is given by the formula  $x \tan \alpha = \frac{v^2}{g}$ .

2. Find the Cartesian equation of the path of a particle projected from the origin with component velocities ( $u$ ), ( $v$ ) along the axes, the  $y$  axis being taken vertically upwards. If a particle is projected so as just to clear a wall of height ( $b$ ) at a horizontal distance ( $a$ ), and to have a range ( $c$ ) from the point of projection, show that the velocity of projection ( $V$ ) is given by

$$2 \frac{V^2}{g} = \frac{a^2(c-a)^2 + b^2c^2}{ab(c-a)}.$$

3. A gun is pointed in a direction  $20^\circ$  W. of N., elevated to  $70^\circ$  and a shot is fired with speed 1500 ft. per sec. and hits an aeroplane which at the instant of firing was 1 mile N. of the gun and travelling horizontally westward. Find the height and speed (assumed uniform) of the aeroplane (neglect resistances).

4. A gun is fired at an elevation  $\alpha$  from the top of a cliff of height  $h$ , the muzzle velocity of the projectile being that due to a height  $h'$ . Obtain the quadratic giving the values of  $\tan \alpha$  corresponding to a range  $R$  on the plain at the foot of the cliff. Prove that the maximum value of  $R$  is  $2\sqrt{h'(h+h')}$  and that for this value of  $R$ ,  $\tan \alpha = \sqrt{h'/(h+h')}$ .

5. A gun which can fire a projectile with an initial velocity  $u$  ft. per sec. is on a plane inclined at an angle  $\alpha$  to the horizon. Prove that the portion of the plane within range of the gun is an ellipse and express the area in acres.

6. A particle is projected under gravity from A so as to pass through B. Show that for a given velocity of projection there are two paths.

Show that if B has horizontal and vertical co-ordinates  $x$ ,  $y$  referred to A, and the velocity of projection is  $\sqrt{2gh}$ , the angle between the two paths at B is a right angle if B lies on the ellipse  $x^2 + 2y^2 = 2hy$ .

7. A shot is fired from O to pass through P and Q. Show that the time from P to Q is  $\sqrt{2c(\tan \alpha - \tan \beta)/g}$ , where  $c$  is the length of the projection of PQ on a horizontal plane and  $\alpha, \beta$  are the angles subtended at O by the vertical lines from P and Q to the horizontal plane through O.

8. From a gun placed on a horizontal plane, which can fire a shell with velocity  $\sqrt{2gH}$ , it is required to throw a shell over a wall of height  $h$ , and the elevation of the gun cannot exceed  $\alpha$ , where  $\alpha < 45^\circ$ . Show that this will be possible only when  $h < H \sin^2 \alpha$ , and that, if this condition be satisfied, the gun must be fired from within a strip of the plane whose breadth is  $4 \cos \alpha \sqrt{H(H \sin^2 \alpha - h)}$ .

9. A heavy particle makes complete revolutions in a smooth circular tube placed in a vertical plane. If the velocity at the lowest point is  $n$  times the velocity at the highest point, show that the pressure of the tube on the particle when moving vertically bears to the pressure at the lowest point the ratio  $2n^2 + 2 : 5n^2 - 1$ .

10. A particle is dropped from a height  $h$  on to a plane inclined at an angle  $\alpha$  to the horizon. How far down the plane is its next point of impact, the co-efficient of restitution being  $e$ ?

11. A smooth wedge ABC of mass  $M$  and angle  $ABC = \alpha$  is placed with the face AB on a smooth horizontal plane. A particle of mass  $m$  moving with velocity  $u$  inclined at  $(\alpha - \beta)$  to the horizon strikes the face BC at an angle  $\beta$ . If  $e$  is the co-efficient of restitution show that the horizontal velocity of the particle after impact will exceed that of the wedge if  $(M + m)(\cos \alpha \cos \beta - e \sin \alpha \sin \beta)$  is greater than  $m \cos^2 \alpha \cdot \cos (\alpha - \beta)$ .

12. A string with heavy particles at the ends lies over a smooth fixed pulley. In the position of equilibrium each particle is level with the centre of the pulley. One particle is displaced downwards so that the system moves under gravity. Find the pressure exerted by the second particle on the pulley as it passes over the highest point, and prove that it leaves the pulley when it has traversed an arc of about  $108\frac{1}{2}^\circ$ .

13. A pendulum is suspended from the roof of a railway carriage travelling at speed  $V$  round a curve of radius  $a$ . It makes  $n$  oscillations per second but when the carriage is stationary the number is

$$n_1. \quad \text{Prove that} \quad V^2 = ag \sqrt{\frac{n^4}{n_1^4} - 1}.$$

14. The range of a rifle bullet is 1200 yards when  $\alpha$  is the elevation of projection. Show that if the rifle is fired with the same elevation from a car travelling at 10 miles per hour towards the target the range will be increased by  $220 \tan^2 \alpha$  feet.

15. A particle is projected along the inner side of a smooth vertical circle of radius  $a$ , the velocity of the lowest point being  $u$ . Show that if  $u^2 < 5ga$  the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is  $2(u^2 - 2ga)^3/27g^3a^2$ .

16. An aeroplane is flying with constant velocity  $v$  and at constant height  $h$ . Show that, if a gun is fired point blank at the aeroplane after it has passed directly over the gun and when its angle of elevation as seen from the gun is  $\alpha$ , the shell will hit the aeroplane provided  $2(V \cos \alpha - v) v \tan^2 \alpha = gh$ , where  $V$  is the initial velocity of the shot, the path being assumed to be parabolic.

17. A particle slides down the surface of a smooth fixed sphere of radius  $a$  starting from rest at the highest point. Find where it will leave the sphere, and show that it will afterwards describe a parabola of latus rectum  $\frac{1}{2}ga$ , and that it will strike the horizontal plane through the lowest point of the sphere at a distance  $5(\sqrt{5} + 4\sqrt{2})a$  from the vertical diameter.

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18. A particle starts from rest at a point  $P$  in the arc of a smooth cycloid whose axis is vertical and vertex  $A$  downwards. The particle is afterwards projected from  $P$  downwards along the curve with velocity equal to that with which it reaches  $A$  when starting from rest at  $P$ . Show that it will now reach  $A$  in half the time taken in the preceding case.

19. A smooth straight tube  $BAC$  is bent at  $A$  and is fixed in a vertical plane so that  $AB, AC$  make angles  $\alpha, \beta$  on opposite sides of the vertical. A heavy uniform string  $PAQ$  in the tube is slightly displaced from the position of equilibrium; show that when a length  $x$  has passed over  $A$  in the direction of  $P$  both the velocity and the acceleration vary as  $x$ , and that the tension of the string at  $A$  varies as  $(p+x)(q-x)$  where  $p, q$  are the lengths  $AP, AQ$  in the position of equilibrium.

20. Two particles are projected from the same point with velocities  $v, v'$  at angles  $\alpha, \alpha'$  respectively with the horizontal; the paths lie in the same vertical plane and the second particle is projected at a time  $T$  after the first. Show that, certain conditions being satisfied, the particles will strike one another before reaching the horizontal plane through the point of projection if

$$T = \frac{2vv' \sin(\alpha - \alpha')}{g(v \cos \alpha + v' \cos \alpha')}.$$

21. A particle suspended by a vertical string is projected horizontally, and after describing an arc of a circle the string slackens and the particle moving in a parabolic path returns to the point of projection. Prove that the inclination of the string to the vertical when it first becomes slack is given by

$$2 \cos^3 \phi + 3 \cos^2 \phi - 1 = 0.$$

22. If a projectile is describing a parabola under gravity, air resistance being neglected, prove that the velocity at any point  $P$  is  $\sqrt{2gc}$ , where  $c$  is the depth of  $P$  below the directrix of the parabola.

Also prove that the least speed with which a particle can be projected from a point  $Q$  on the ground so that it may pass over a vertical wall of height  $h$  at a distance  $d$  from  $Q$  is, if air resistance be neglected,  $\sqrt{g(h + \sqrt{h^2 + d^2})}$ .



23. A particle slides down a smooth wire in the form of a cycloid whose axis is vertical and vertex downwards. Prove that the projection of the particle on a vertical line moves with simple harmonic motion. Also prove that when the particle starts from rest at the cusp, its acceleration is constant in magnitude and directed towards a point moving uniformly in a horizontal line.

24. A particle of mass  $m$  is suspended from a fixed point by a string of length  $a$ . Another string attached to the particle  $m$  passes through a ring fixed at a distance  $2a$  vertically below the fixed point and has a mass  $M$  attached to its other end. If the mass  $m$  is slightly displaced from the position of equilibrium, prove that the

time of oscillation of the system is  $2\pi \sqrt{\frac{a}{g} \cdot \frac{m}{m+2M}}$ .

25. Prove that when a particle slides from rest down a smooth cycloid, with axis vertical and vertex downwards, the time of reaching the vertex is independent of the initial position.

Two particles slide down the cycloid, starting simultaneously from rest. Show that the vertical component of the motion of the middle point of the chord joining them is a simple harmonic motion.

26. A particle starts from rest at a given point of a smooth cycloid whose vertex is downwards and base horizontal. Prove that it falls through  $1/n$  of the vertical distance to the lowest point

in the time  $2\sqrt{\frac{a}{g}} \sin^{-1} \frac{1}{\sqrt{n}}$ .

27. A cycloid has its base horizontal and vertex upwards; prove that the time of falling down any radius of curvature is constant.

28. If a particle of weight  $w$  is suspended from the roof of a railway carriage which is moving with a constant speed,  $v$ , in a curve, prove that in the position in which the particle is at rest relatively to the carriage, the tension of the cord is

$$w \left( 1 + \frac{v^4}{g^2 \rho^2} \right)^{\frac{1}{2}}.$$

29. Two equal particles are tied together by a light string of length  $\frac{\pi a}{2}$  and rest in equilibrium on the surface of a smooth circular cylinder in a plane perpendicular to the axis of the cylinder, which is horizontal, the radius of the cylinder being  $a$ . The particles are then slightly displaced in a plane perpendicular to the axis of the cylinder. Prove that when the lower of the two particles leaves the surface, the pressure on the other is rather more than three-fourths of its weight.

30. A parabolic trajectory passes through two given points, in directions inclined at angles  $(\alpha, \beta)$  to the horizontal. Show that for all such trajectories  $(\tan \alpha + \tan \beta)$  has the same value.

31. A particle of mass 10 grams is set sliding down the smooth curve  $y = \frac{x^2}{5}$  which is in a vertical plane, from the point  $x = 10$ . Find an expression for (i.) the normal pressure, (ii.) the kinetic energy, (iii.) the tangential acceleration of any point  $x$  and evaluate when  $x = 5$ .

32. A particle P of unit mass, free to slide on a smooth wire, is attracted towards a point O of the wire with a force  $\mu \cdot OP$ . If the wire be made to revolve about O in a horizontal plane with constant angular velocity  $\omega$ , show that the motion of P on the wire is a simple harmonic motion of period  $2\pi/\sqrt{\mu - \omega^2}$  and that when  $\omega^2 = \frac{1}{2}\mu$  the path of P in space is a circle.

33. A smooth semi-circular tube, radius  $a$ , has its diameter horizontal and rotates with uniform angular velocity  $\omega$  about a vertical radius. A particle is placed in one end of the tube and allowed to slide down the tube. Prove that if  $a\omega^2 > 2g$ , the particle will return to the initial position after descending to a depth  $\frac{2g}{a\omega^2}$ . Also prove that the time taken is given by

$$2\sqrt{a} \int_0^{\alpha} \frac{d\theta}{\sqrt{\sin \theta (2g - a\omega^2 \sin \theta)}} \quad \text{where } \sin \alpha = \frac{2g}{a\omega^2}.$$

34. Find the law of force parallel to the axis of  $y$  under the action of which a particle will describe the curve  $x^2 + ay^2 = bx$ .

35. Find the law of force which acting parallel to the line  $x - y = 0$  causes a particle to describe the curve  $x^3 + y^3 = a^3$ .

36. Over a small smooth pulley is placed a uniform flexible chain; the latter is allowed to move from rest when lengths  $l - a$  and  $l + a$  hang down on the two sides. Show that the chain will leave the pulley after a time  $\sqrt{\frac{l}{g}} \cosh^{-1} \frac{l}{a}$ .

37. A particle of mass  $m$ , moves in a smooth circular tube of radius  $a$ , under the action of a force  $m\mu \times \text{distance}$ , to a point inside the tube at a distance  $d$  from its centre. If the particle be placed very nearly at its greatest distance from the centre of force, show that it will describe the quadrant ending at its least distance in time  $\sqrt{\frac{a}{\mu d}} \log(\sqrt{2} + 1)$ . Also show that when the particle has described from rest an arc subtending an angle  $\theta$  at the centre of the tube, the reaction is  $\mu m [a + d(3 \cos \theta - 2)]$ .

38. A particle of mass  $m$  slides down the rough inclined face of a wedge of mass  $M$  and inclination  $\alpha$ , which is free to move on a smooth horizontal plane. Show that the time of describing any distance from rest is less than the time taken when the wedge is fixed in the ratio

$$\left\{ 1 - \frac{m \cos \alpha (\cos \alpha + \mu \sin \alpha)}{M + m} \right\}^{\frac{1}{2}} : 1.$$

39. Two particles of equal mass are attached to the ends of a light rod. The rod can turn freely about a point O at distances  $l$  and  $l'$  respectively from the ends. Prove that if the rod rotate about a vertical axis through O with angular velocity  $\omega$ , it is inclined to the vertical at an angle

$$\cos^{-1} \left\{ \frac{(l-l')g}{\omega^2(l^2+l'^2)} \right\}.$$

40. Two small rings of equal mass can slide on a smooth parabolic wire with axis vertical and vertex upwards. The rings are connected by an elastic string equal in length to the latus rectum, the modulus of elasticity being equal to four times the weight of either ring. If the rings slide down the two sides of the parabola starting from rest at the vertex, show that they will come to rest at a depth equal to the latus rectum.

41. A smooth wedge of angle  $\alpha$  is at rest on a smooth horizontal plane. The mass of the wedge is  $M$ . A particle of mass  $m$  falls vertically and strikes the inclined face of the wedge, the magnitude of its velocity at the instant of impact being  $V$ . The coefficient of restitution between the particle and the wedge is  $e$ . Show that after the impact the wedge moves with a speed

$$\frac{mV(1+e)\cos\alpha\sin\alpha}{M+m\sin^2\alpha}$$

and find the components of the velocity of the particles, just after impact, along and perpendicular to the inclined face of the wedge. If the particle, after rebounding, strikes the moving wedge again, show that the distance between the two points of impact, measured along the plane, is

$$\frac{2eV^2(M+m)(1+e)\sin\alpha}{g(M+m\sin^2\alpha)}.$$

42. A bead of mass  $m$ , which can slide on a smooth horizontal rail, is connected by a light rod of length  $l$  to a particle of equal mass. If the rod is released from a horizontal position in the vertical plane through the rail, show that when it has turned through an angle  $\theta$  the velocity of the bead is  $\frac{1}{2}l\dot{\theta}\sin\theta$ . Find  $\dot{\theta}$  in terms of  $\theta$ , and show that when  $\theta = \frac{1}{2}\pi$  the tension in the rod is  $5mg$ .

## CHAPTER V

### CENTRES OF GRAVITY : ATTRACTIONS

**31. Centre of Gravity of a System of Particles in One Plane.**—Let  $m_1, m_2, \dots$  be the masses of the particles and  $(x_1, y_1), (x_2, y_2), \dots$  the co-ordinates referred to rectangular axes; then if  $(\bar{x}, \bar{y})$  be the co-ordinates of the centre of gravity or centre of inertia of the system,

$$(m_1 + m_2 + \dots) \bar{x} = m_1 x_1 + m_2 x_2 + \dots$$

that is 
$$\bar{x} = \frac{\sum mx}{\sum m}.$$

Similarly 
$$\bar{y} = \frac{\sum my}{\sum m}.$$

### 32. C.G. of a Thin Wire.

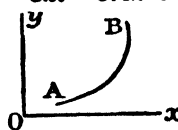
 AB represents the wire: let  $(x_1, y_1)$  and  $(x_2, y_2)$  be the co-ordinates of A and B respectively and let  $m$  be the mass per unit length. Then mass of an element  $= m \delta s$

Fig. 49.

$$\therefore \bar{x} = \frac{\int mx ds}{\int m ds}, \text{ the limits being from the value of } s$$

at A to that at B;

$$= \frac{\int_{x_1}^{x_2} mx \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dx}{\int_{x_1}^{x_2} m \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} dx}.$$

$$\text{Similarly } \bar{y} = \frac{\int_{y_1}^{y_2} my \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} dy}{\int_{y_1}^{y_2} m \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} dy}.$$

The cartesian equation of the curve taken up by the wire being known,  $\frac{dy}{dx}$  and  $\frac{dx}{dy}$  can be expressed in terms of  $x$  and  $y$  respectively. If  $m$  is a constant, it can be taken out from under the sign of integration and will then cancel. If  $m$  be a variable it must be expressed as a function of  $x$  in the expression for  $\bar{x}$  and as a function of  $y$  in the expression for  $\bar{y}$ .

If the polar equation to the curve be given, and  $x\hat{O}A = \alpha$  and  $x\hat{O}B = \beta$ , then as  $\delta s = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} \delta\theta$ ,

$$\bar{x} = \frac{\int_{\alpha}^{\beta} m r \cos \theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta}{\int_{\alpha}^{\beta} m \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta}$$

and  $\bar{y} = \frac{\int_{\alpha}^{\beta} m r \sin \theta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta}{\int_{\alpha}^{\beta} m \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta}.$

**Ex. I.** Find the position of the centroid of

- (1) an arc of a circle, radius  $a$  subtending angle  $2\alpha$  at the centre,
- (2) an arc of a cycloid (radius of generating circle,  $a$ ) from vertex to cusp.

(1) By symmetry the centroid of the circular arc will lie on the line  $Ox$  which bisects the angle between the radii.

Let  $P\hat{O}x = \theta$  and  $P\hat{O}Q = \delta\theta$ , where  $PQ$  is a small portion of the arc; then mass of element  $= m a \delta\theta$ .

$$\therefore \bar{x} = \frac{2 \int_0^{\alpha} m \cdot a \cos \theta \cdot a d\theta}{2a a \cdot m} = \frac{a \int_0^{\alpha} \cos \theta \cdot d\theta}{a} = \frac{a \sin \alpha}{a}.$$

**NOTE 1.** For a semi-circular arc,  $\bar{x} = \frac{2a}{\pi}$ .

(2) Take the vertex  $O$  as origin, and the tangent at  $O$  as the axis of  $x$ .

The equation of the cycloid is  $s = 4a \sin \psi$ , where  $s$  is measured from  $O$  and  $\psi$  is the angle a tangent makes with  $Ox$ .

For any element  $PQ$ ,

$$\begin{aligned} \delta s &= 4a \cos \psi \cdot \delta\psi, \\ x \delta s &= a(2\psi + \sin 2\psi) \cdot 4a \cos \psi \cdot \delta\psi \\ &= 4a^2 \left[ 2\psi \cos \psi + \frac{1}{2} (\sin 3\psi + \sin \psi) \right] \delta\psi. \end{aligned}$$

$$\begin{aligned} \text{Now } \int \psi \cos \psi d\psi &= \psi \sin \psi - \int \sin \psi \cdot d\psi \\ &= \psi \sin \psi + \cos \psi. \end{aligned}$$

$$\therefore \bar{x} = \frac{4a^2 \left[ 2\psi \sin \psi + 2 \cos \psi - \frac{1}{2} \cos 3\psi - \frac{1}{2} \cos \psi \right]_0^{\frac{\pi}{2}}}{4a \left[ \sin \psi \right]_0^{\frac{\pi}{2}}}.$$

$$= \frac{a \left[ \pi - (2 - \frac{1}{2} - \frac{1}{2}) \right]}{1} = \frac{a}{3} (3\pi - 4).$$

$$\begin{aligned} \text{Also } y\delta s &= a(1 - \cos 2\psi) \cdot 4a \cos \psi \cdot \delta\psi. \\ &= 4a^2 \left[ \cos \psi - \frac{1}{2} (\cos 3\psi + \cos \psi) \right] \delta\psi. \end{aligned}$$

$$\therefore \bar{y} = a \left[ \sin \psi - \frac{1}{2} \sin 3\psi - \frac{1}{2} \sin \psi \right]_0^{\frac{\pi}{2}} = \frac{2}{3}a.$$

### 33. C.G. of a Plane Lamina.

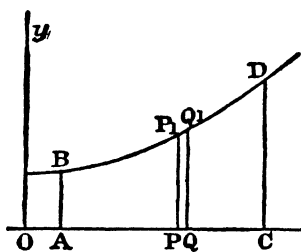


Fig. 50.

BD is a curve whose cartesian equation is known. It is required to find the co-ordinates of the C.G. of a lamina ACDB, where AB and CD are ordinates and OA = a, OC = c.

Mass of element PQQ<sub>1</sub>P<sub>1</sub> = myδx, where m is the mass per unit area of the element.

$$\therefore \bar{x} = \frac{\int_a^c mxy dx}{\int_a^c my dx}.$$

If the density of the element PQQ<sub>1</sub>P<sub>1</sub> be uniform, its C.G. is at a distance  $\frac{y}{2}$  from the x axis

$$\therefore \bar{y} = \frac{\frac{1}{2} \int_a^c m \cdot y^2 dx}{\int_a^c my dx}.$$

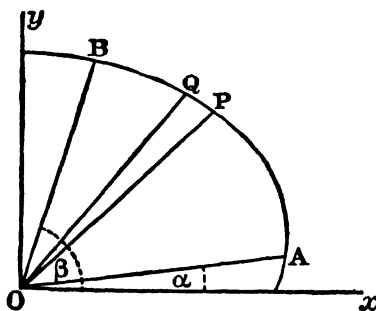


Fig. 51.

tance  $\frac{2}{3}r$  from O.

AB is a curve whose polar equation is known. It is required to find the co-ordinates of the C.G. of a lamina bounded by the curve, OA and OB where  $\angle AOA = \alpha$  and  $\angle BOA = \beta$ .

If the density of the element POQ is uniform, m per unit area, the mass of the element will be  $m \cdot \frac{1}{2} r^2 \delta\theta$  and its C.G. will be at a distance

$$\therefore \bar{x} = \frac{\int_a^\beta \frac{1}{2} \cdot \frac{1}{2} m r^2 \cos \theta d\theta}{\int_a^\beta \frac{1}{2} m r^2 d\theta} \quad \text{and} \quad \bar{y} = \frac{1}{2} \cdot \frac{\int_a^\beta m r^2 \sin \theta d\theta}{\int_a^\beta m r^2 d\theta}.$$

NOTE 2. If the density of element  $PQQ_1P_1$  be not uniform, double integration notation can be avoided by first obtaining the mass and the ordinate of its C.G. For example, suppose the density of the lamina  $ACDB$  varies as the distance from  $Ox$  (i.e.  $ky$ ),

$$\text{then mass of element } PQQ_1P_1 = \int_0^y k y \delta x dy = \frac{1}{2} k y^2 \delta x$$

$$\text{and distance of its C.G. from } Ox = \frac{\int_0^y k y^2 \delta x dy}{\frac{1}{2} k y^2 \delta x} = \frac{2}{3} y$$

$$\therefore \bar{y} = \frac{\frac{1}{2} \int_a^c y^3 dx}{\frac{1}{2} \int_a^c y^2 dx}$$

In a similar way, if the density of the lamina  $OAPB$  is a function of the distance from  $O$ , the mass of the element  $POQ$  and the distance of its C.G. from  $O$  can be first obtained and thus  $\bar{x}$  and  $\bar{y}$  evaluated.

**Ex. II.** Find the position of the C.G. of a lamina of uniform density in the shape of a sector of a circle radius  $a$ , angle  $2\alpha$ .

*Method I.*

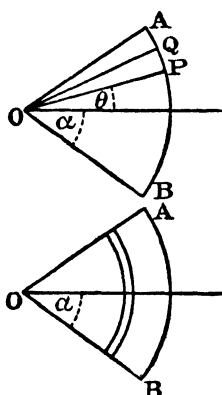


Fig 52

By symmetry, the C.G. lies on  $Ox$ , which bisects  $\widehat{AOB}$ .

Let  $m$  be the mass per unit area, then mass of element  $POQ = \frac{1}{2} a^2 \delta \theta m$ .

$$\therefore \bar{x} = \frac{2 \int_0^\alpha \frac{1}{2} m a^2 \cdot \frac{1}{2} a \cos \theta d\theta}{a a^2 m} = \frac{2a}{3a} \sin \alpha.$$

*Method II.*

Mass of element consisting of an arc, radius  $x$  and breadth  $\delta x = 2\alpha x \delta x \cdot m$ .

$$\text{Distance of its C.G. from } O = \frac{x \sin \alpha}{a}$$

$$\therefore \bar{x} = \frac{\int_0^a 2m x^2 \sin \alpha dx}{a a^2 m} = \frac{\frac{2a^3 \sin \alpha}{3}}{a a^2} = \frac{2a}{3a} \sin \alpha.$$

For a semi-circle,  $\alpha = \frac{\pi}{2}$  and thus  $\bar{x} = \frac{4a}{3\pi}$ .

**Ex. III.** Find the position of the C.G. of a semi-circular lamina radius  $a$  whose surface density varies as

- (i.) the square of the distance from the diameter,  
 (ii.) the square of the distance from the middle point of the diameter.

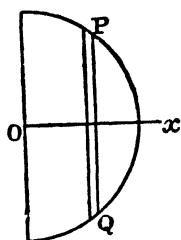


Fig. 53.

- (i) Let the density be  $kx^2$ .

Then mass of element PQ

$$= 2\sqrt{(a^2 - x^2)} \cdot \delta x \cdot kx^2.$$

$$\bar{x} = \frac{2k \int_0^a x^3 \sqrt{(a^2 - x^2)} dx}{2k \int_0^a x^2 \sqrt{(a^2 - x^2)} dx}.$$

Let  $x = a \sin \theta$ , so that  $\delta x = a \cos \theta \delta \theta$ ,

$$\begin{aligned} \text{then } \int_0^a x^3 \sqrt{a^2 - x^2} dx &= a^5 \int_0^{\frac{\pi}{2}} \sin^3 \theta \cdot \cos^2 \theta d\theta \\ &= a^5 \int_0^{\frac{\pi}{2}} (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta. \end{aligned}$$

$$\text{Also } \int_0^a x^2 \sqrt{a^2 - x^2} dx = a^4 \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin^2 2\theta d\theta = \frac{a^4}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 4\theta) d\theta,$$

$$\begin{aligned} \therefore \bar{x} &= \frac{8a \left[ -\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta \right]_0^{\frac{\pi}{2}}}{\left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\frac{\pi}{2}}} = \frac{8a \left( \frac{1}{3} - \frac{1}{5} \right)}{\frac{\pi}{2}} = \frac{32a}{15\pi}. \end{aligned}$$

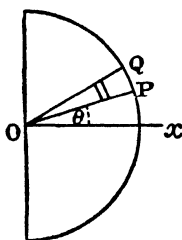


Fig. 54.

- (ii) Let the density be  $kr^2$ .

Then mass of element POQ

$$= \int_0^a \delta \theta \cdot r \cdot dr \cdot kr^2 = k\delta \theta \cdot \frac{a^4}{4}$$

and distance of its C.G. from O

$$= \frac{\int_0^a \delta \theta \cdot r dr \cdot kr^2 \cdot r}{k\delta \theta \cdot \frac{a^4}{4}} = \frac{4a}{5}.$$

$$\begin{aligned} \therefore \bar{x} &= \frac{2 \int_0^{\frac{\pi}{2}} \frac{1}{8} ka^5 \cos \theta d\theta}{2 \int_0^{\frac{\pi}{2}} \frac{1}{4} ka^4 d\theta} = \frac{4a}{5} \cdot \frac{1}{\frac{\pi}{2}} = \frac{8a}{5\pi}. \end{aligned}$$



**34. C.G. of a Pyramid.**—Let  $O$  be the vertex of the pyramid and  $C$ , the C.G. of the lamina forming the base. Then  $\bar{G}$ , the C.G. of the pyramid, will lie on the line  $OC$ .

Let  $\bar{x}$  be the distance of  $G$  from the vertex measured along the perpendicular to the base and let the thickness of an element at distance  $x$  from the vertex and parallel to it be  $\delta x$ . Let  $A$  be the area of the base,  $h$  the perpendicular height of the pyramid and  $m$  the mass per unit volume.

Then mass of element  $= A \frac{x^2}{h^3} m \delta x$ ,

$$\therefore \bar{x} = \frac{\frac{Am}{h^3} \int_0^h x^3 dx}{\frac{Am}{h^3} \int_0^h x^2 dx} = \frac{\frac{h^4}{4}}{\frac{h^3}{3}} = \frac{3}{4}h.$$

**35. Centroid of a Surface of Revolution.**—Referring to figure 50, suppose the arc  $BD$  rotates about  $Ox$ , then a surface is generated. Let  $m$  be the mass per unit area of this surface.

By symmetry, the centroid of this surface lies on  $Ox$ .

Mass of element formed by rotation of  $P_1Q_1 = 2\pi y \delta s \cdot m$

$$\therefore \bar{x} = \frac{\int_a^c m x y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx}{\int_a^c m y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx}.$$

Referring to figure 51,

mass of element formed by rotation of  $PQ = 2\pi r \sin \theta \cdot \delta s \cdot m$ ,

$$\therefore \bar{x} = \frac{\int_a^\beta m r^2 \sin \theta \cos \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta}{\int_a^\beta m r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta}.$$

**Ex. IV.** Find the position of the centroid of the portion of the area of a sphere, radius  $a$ , lying between parallel planes at distances  $b$  and  $c$  from the centre.

Mass of element formed by rotation of  $PQ$  about  $Ox = 2\pi a \sin \theta \cdot a \delta \theta \cdot m$ .

Let  $\widehat{xOB} = \beta$  and  $\widehat{xOC} = \gamma$ .

$$\text{Then } \bar{x} = \frac{a \int_\gamma^\beta \sin \theta \cos \theta d\theta}{\int_\gamma^\beta \sin \theta d\theta}$$

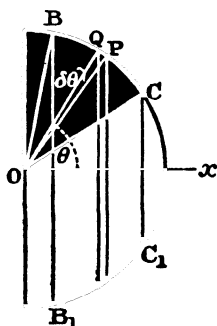


Fig. 55.

$$\begin{aligned}
 &= \frac{a \left[ -\frac{1}{2} \cos^2 \theta \right]_{\gamma}^{\beta}}{\left[ -\cos \theta \right]_{\gamma}^{\beta}} = \frac{a}{2} \cdot \frac{\cos^2 \gamma - \cos^2 \beta}{\cos \gamma - \cos \beta} = \frac{a}{2} (\cos \gamma + \cos \beta) \\
 &= \frac{a}{2} \cdot \left( \frac{c}{a} + \frac{b}{a} \right) = \frac{1}{2} (b + c).
 \end{aligned}$$

For a thin hemispherical shell,  $b = 0$  and  $c = a$   $\therefore \bar{x} = \frac{a}{2}$ .

**Ex. V.** Find the position of the centroid of the portion of the surface formed by rotation of the curve  $y^2 = 4ax$ , and bounded by the planes  $x = 0$  and  $x = a$ .

The centroid lies on  $Ox$  at a distance from  $O$  given by

$$\begin{aligned}
 \bar{x} &= \frac{\int_0^a xy \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx}{\int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx} = \frac{\int_0^a 2a x^{\frac{1}{2}} \sqrt{1 + \frac{a}{x}} \cdot dx}{\int_0^a 2a x^{\frac{1}{2}} \sqrt{1 + \frac{a}{x}} \cdot dx} \\
 &= \frac{\int_0^a x \sqrt{x+a} \cdot dx}{\int_0^a \sqrt{x+a} \cdot dx} = \frac{\left[ \frac{2}{3} x(x+a)^{\frac{3}{2}} - \frac{4}{15} (x+a)^{\frac{5}{2}} \right]_0^a}{\left[ \frac{2}{3} (x+a)^{\frac{3}{2}} \right]_0^a} \\
 &= \frac{2(\sqrt{2} + 1)}{5(2\sqrt{2} - 1)} \cdot a.
 \end{aligned}$$

**36. C.G. of a Volume of Revolution.**—Referring to figure 50, suppose the area  $ACDB$  to rotate about  $Ox$ , then a volume is generated.

Let  $m$  be the mass per unit volume.

By symmetry, the C.G. lies on  $Ox$ .

Mass of element formed by rotation of  $PQQ_1P_1 = \pi y^2 \cdot \delta x \cdot m$

$$\therefore \bar{x} = \frac{\int_a^c m x y^2 dx}{\int_a^c m y^2 dx}.$$

**Ex. VI.** Find the position of the C.G. of the portion of a sphere (radius  $a$ ) of uniform density ( $\rho$ ), lying between parallel planes at distances  $b$  and  $c$  from the centre.

Referring to figure 55,

mass of element  $= \pi y^2 \delta x \cdot \rho$ .

$$= \pi(a^2 - x^2) \delta x \cdot \rho.$$

$$\begin{aligned} \therefore \bar{x} &= \frac{\pi\rho \int_b^c (a^2x - x^3) dx}{\pi\rho \int_b^c (a^2 - x^2) dx} = \frac{\left[ \frac{1}{2}a^2x^2 - \frac{1}{4}x^4 \right]_b^c}{\left[ a^2x - \frac{1}{3}x^3 \right]_b^c} \\ &= \frac{3(c+b)(2a^2 - c^2 - b^2)}{4(3a^2 - c^2 - cb - b^2)}. \end{aligned}$$

For a hemisphere,  $b = 0$  and  $c = a$ ,  $\therefore \bar{x} = \frac{3a}{8}$ .

**Ex. VII.** Find the position of the C.G. of a hemisphere (radius  $a$ ) whose density varies as the square of the distance from the centre of the complete sphere.

Suppose the hemisphere to be divided into a number of thin hemispherical shells, then

$$\text{mass of element of radius } x = 2\pi x^2 \cdot \delta x \cdot kx^2$$

and distance of C.G. of element from the centre of the base =  $\frac{x}{2}$ .

$$\therefore \bar{x} = \frac{k\pi \int_0^a x^5 dx}{2k\pi \int_0^a x^4 dx} = \frac{\frac{a^6}{6}}{\frac{2a^5}{5}} = \frac{5a}{12}.$$

**Ex. VIII.** Find the position of the C.G. of a sphere (radius  $a$ ) whose density varies as the distance from a tangent plane.

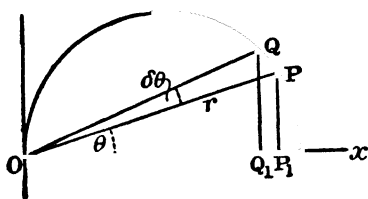


Fig. 56.

Taking the diameter through the point of contact of the plane as the axis of  $x$ , then a sphere is obtained by rotation about  $Ox$  of the semicircle bounded by the curve whose polar equation is  $r = 2a \cos \theta$ . The limits will

obviously be  $\theta = 0$  and  $\theta = \frac{\pi}{2}$ .

Now, as

$$x = 2a \cos^2 \theta, \quad y = 2a \cos \theta \sin \theta \quad \text{and} \quad \delta x = -4a \cos \theta \sin \theta \cdot \delta \theta,$$

$$\text{mass of element obtained by rotation of } PP_1Q_1Q = \pi y^2 \delta x \cdot kx$$

$$= -\pi \cdot 32 a^4 k \cos^5 \theta \cdot \sin^3 \theta \cdot \delta \theta$$

$$\therefore \bar{x} = \frac{2a \int_0^{\frac{\pi}{2}} \cos^7 \theta \sin^3 \theta d\theta}{\int_0^{\frac{\pi}{2}} \cos^5 \theta \sin^3 \theta d\theta} = \frac{2a \int_0^{\frac{\pi}{2}} (\cos^7 \theta - \cos^9 \theta) \sin \theta d\theta}{\int_0^{\frac{\pi}{2}} (\cos^5 \theta - \cos^7 \theta) \sin \theta d\theta}$$

$$= \frac{2a \left[ -\frac{1}{3} \cos^3 \theta + \frac{1}{15} \cos^{15} \theta \right]_0^{\frac{\pi}{2}}}{\left[ -\frac{1}{3} \sin^3 \theta + \frac{1}{15} \cos^5 \theta \right]_0^{\frac{\pi}{2}}} = \frac{6a}{5}.$$

**Ex. IX.** ABC is an arc of a circle, centre O and radius  $a$ .

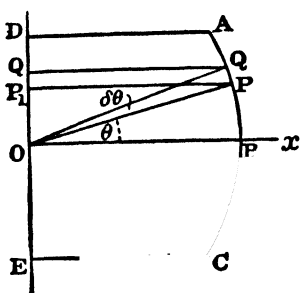


Fig. 57.

$\widehat{AOB} = \widehat{COB} = \alpha$  and AD, CE are perpendicular to Oy.

DABCE rotates through an angle  $\pi$  about Oy, the plane of the paper being the mean position. Find the C.G. of the solid generated.

By symmetry, the C.G. will lie on Ox.

Vol. of element obtained by rotation of  $PP_1Q_1Q$

$$= \frac{\pi}{2} \cdot a^2 \cos^2 \theta \cdot a \cos \theta \cdot \delta \theta.$$

Also, distance of its C.G. from Oy =  $\frac{4a \cos \theta}{3\pi}$ .

$$\begin{aligned} \therefore \bar{x} &= \frac{4a}{3\pi} \cdot \frac{2 \int_0^{\alpha} \cos^4 \theta d\theta}{2 \int_0^{\alpha} \cos^2 \theta d\theta} = \frac{4a}{3\pi} \cdot \frac{\frac{1}{8} \int_0^{\alpha} (3 + 4 \cos 2\theta + \cos 4\theta) d\theta}{\int_0^{\alpha} (1 - \sin^2 \theta) \cos \theta d\theta} \\ &= \frac{4a}{3\pi} \cdot \frac{3\alpha + 2 \sin 2\alpha + \frac{1}{4} \sin 4\alpha}{8[\sin \alpha - \frac{1}{8} \sin^3 \alpha]}. \end{aligned}$$

**Ex. X.** Referring to figure 57, if ABOD rotate about Ox, find the position of the C.G. of the solid generated.

By symmetry, the C.G. will lie on Oy.

$PP_1Q_1Q$ , when rotated, generates an element consisting of a thin hollow cylinder.

Vol. of element =  $2\pi \cdot a \sin \theta \cdot a \cos \theta \cdot a \cos \theta \cdot \delta \theta$ .

Dist. of C.G. of element from O =  $\frac{1}{2} a \cos \theta$ .

$$\therefore \bar{y} = \frac{a}{2} \cdot \frac{\int_0^{\alpha} \sin \theta \cos^3 \theta d\theta}{\int_0^{\alpha} \sin \theta \cos^2 \theta d\theta} = \frac{a}{2} \cdot \frac{\left[ -\frac{1}{4} \cos^4 \theta \right]_0^{\alpha}}{\left[ -\frac{1}{3} \cos^3 \theta \right]_0^{\alpha}} = \frac{3a}{8} \cdot \frac{1 - \cos^4 \alpha}{1 - \cos^3 \alpha}.$$

**37. Theorem of Pappus (or Guldin's Theorem).—**

(1) AB is a curve of length L. On being rotated about Oy, an axis which does not intersect the curve, it generates

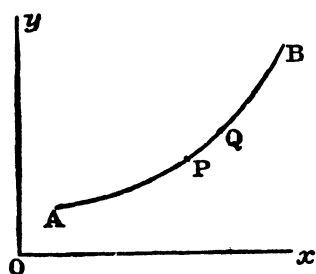


Fig. 58.

a surface whose area is  $2\pi L \bar{x}$ , where  $\bar{x}$  is the distance of the centroid of the arc AB from Oy.

Suppose the arc to be divided up into elements of lengths  $l_1, l_2, \dots$  whose distances from Oy are  $x_1, x_2, \dots$  respectively.

Then, area of surface of revolution  
 $= 2\pi x_1 l_1 + 2\pi x_2 l_2 + \dots$   
 $= 2\pi (l_1 x_1 + l_2 x_2 + \dots)$   
 $= 2\pi \sum l x$

$$= 2\pi L \cdot \bar{x} \quad \text{as} \quad \bar{x} = \frac{\sum l x}{\sum l}.$$

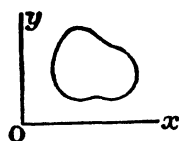


Fig. 59.

(2) A curve, enclosing an area A, rotates about Oy, an axis which does not intersect the area. Then the volume of the ring formed is  $2\pi A \bar{x}$  where  $\bar{x}$  is the distance of the centroid of the area from Oy.

Suppose the area to be divided into elements of area  $a_1, a_2, \dots$  at distances  $x_1, x_2, \dots$  respectively from Oy.

Then volume of ring  $= 2\pi x_1 a_1 + 2\pi x_2 a_2 + \dots$   
 $= 2\pi \cdot \sum a x$

$$= 2\pi A \bar{x} \quad \text{as} \quad \bar{x} = \frac{\sum a x}{\sum a}.$$

**Ex. XI.** Find the area of the surface and the volume of an anchor ring whose mean radius is R, the radius of section being  $r$ . If the ring be cut symmetrically into two portions by a thin hollow cylinder of radius R, find the ratio of the areas of the curved surfaces and the ratio of the volumes.

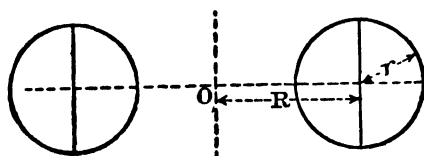


Fig. 60.

Area of surface of ring

$$= 2\pi \cdot 2\pi r \cdot R$$

$$= 4\pi^2 r R.$$

Volume of ring

$$= 2\pi \cdot \pi r^2 R$$

$$= 2\pi^2 r^2 R.$$

$$\frac{\text{Area of curved surface of inner portion}}{\text{Area of curved surface of outer portion}} = \frac{2\pi \cdot \pi r \cdot \left(R - \frac{2r}{\pi}\right)}{2\pi \cdot \pi r \cdot \left(R + \frac{2r}{\pi}\right)}$$

$$= \frac{\pi R - 2r}{\pi R + 2r}.$$



Thus the resultant attraction of the rod CD is the same in magnitude and direction as that of the rod AB.

By symmetry, the resultant attraction of the rod CD is along PE where PE is the bisector of  $\widehat{CPD}$ .

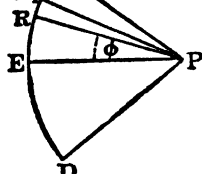


Fig. 62.

Let  $\widehat{EPR} = \phi$ ,  $\widehat{PRR_1} = \delta\phi$ .

Then resultant attraction

$$= \frac{k M m}{p} \int_{-\frac{1}{2}(\alpha + \beta)}^{\frac{1}{2}(\alpha + \beta)} \cos \phi \cdot d\phi.$$

$$= \frac{2 k M m}{p} \sin \frac{1}{2}(\alpha + \beta). = \frac{2 k M m}{p} \sin \frac{\widehat{APB}}{2}.$$

When the rod is infinite in both directions, the resultant attraction is  $\frac{2 k M m}{p}$  in the direction perpendicular to the rod.

**Ex. XIII.** A thin lamina, mass  $m$  per unit area is bent into the shape of the curved surface of half a cylinder radius  $a$  and length  $l$ , so that a section everywhere is a semi-circle. Find the resultant attraction on a particle of unit mass at the mid point of the axis of the cylinder. Also find the resultant attraction of a solid half-cylinder, mass  $\rho$  per unit volume, on the particle of unit mass.

Suppose the surface to be divided into elemental rods of length  $l$  and breadth  $a\delta\theta$ . The mass per unit length of any one of these rods, PQ, is  $ma\delta\theta$ .

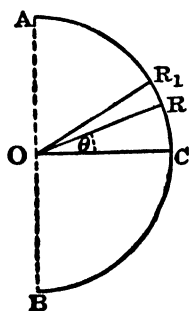


Fig. 63.

Let O be the position of the particle of unit mass, ACB the central section cutting the element PQ at RR<sub>1</sub>. OC is at right angles to the diameter AB and  $\widehat{COR} = \theta$ ,  $\widehat{ROR_1} = \delta\theta$ . Resultant attraction of element PQ

$$= \frac{2 k m a \delta\theta}{a} \sin \frac{\widehat{POQ}}{2}$$

$$= 2 k m \delta\theta \cdot \frac{l}{\sqrt{l^2 + 4a^2}}.$$

The resultant attraction of the lamina acts by symmetry, along OC and its magnitude is

$$\frac{2 k m l}{\sqrt{l^2 + 4a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta \, d\theta = \frac{4 k m l}{\sqrt{l^2 + 4a^2}};$$

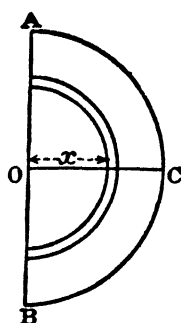


Fig. 64.

Suppose the solid half-cylinder to be divided into thin half-cylindrical shells having a common axis.

An element, radius  $x$  and thickness  $\delta x$  has a mass per unit area of  $\rho \delta x$ .

The resultant attraction of the solid acts, by symmetry, along OC and its magnitude is

$$4kl\rho \int_0^a \frac{dx}{\sqrt{l^2 + 4x^2}} = 2kl\rho \left[ \log(2x + \sqrt{l^2 + 4x^2}) \right]_0^a \\ = 2kl\rho \log \frac{2a + \sqrt{l^2 + 4a^2}}{l}.$$

**Ex. XIV.** Two thin rods AB and CD each of length  $l$  and mass  $m$  per unit length lie parallel to one another at a distance  $2a$  and so that ABDC is a rectangle. Find the resultant attraction of one rod on the other. If a particle be placed in the position of equilibrium and be given a small displacement in the direction parallel to the rods, find the periodic time of the vibrations.

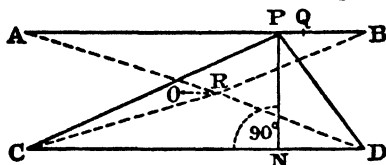


Fig. 65.

Consider an element PQ of the rod AB.

Let  $AP = x$ ,  $PQ = \delta x$ , then the resolved attraction perp. to the rods, of the rod CD on the element PQ is

$$\frac{km^2\delta x}{2a} (\sin \widehat{DPN} + \sin \widehat{NPC}) \\ = \frac{km^2\delta x}{2a} \cdot \left[ \frac{x}{\sqrt{x^2 + 4a^2}} + \frac{(l-x)}{\sqrt{(l-x)^2 + 4a^2}} \right].$$

The resultant attraction of the rod CD on the rod AB is, by symmetry, along the line joining the middle points of the rods and its magnitude is

$$\frac{km^2}{2a} \int_0^l \left[ \frac{x}{\sqrt{x^2 + 4a^2}} + \frac{(l-x)}{\sqrt{(l-x)^2 + 4a^2}} \right] dx \\ = \frac{km^2}{2a} \left[ \sqrt{x^2 + 4a^2} - \sqrt{(l-x)^2 + 4a^2} \right]_0^l = \frac{km^2}{a} \left[ \sqrt{l^2 + 4a^2} - 2a \right].$$

Let O be the position of a particle mass  $M$  when in equilibrium, and R its displaced position. Let  $OR = y$ .

Then

$$M \frac{d^2y}{dt^2} = -2kMm \left( \frac{1}{RB} - \frac{1}{RA} \right)$$



$$\begin{aligned}
 \therefore \frac{d^2 y}{dt^2} &= -2km \left[ \frac{1}{\sqrt{\left(\frac{l}{2} - y\right)^2 + a^2}} - \frac{1}{\sqrt{\left(\frac{l}{2} + y\right)^2 + a^2}} \right] \\
 &= -4km \left[ \{l^2 + 4a^2 - 4y(l-y)\}^{-\frac{1}{2}} - \{l^2 + 4a^2 + 4y(l+y)\}^{-\frac{1}{2}} \right] \\
 &= -\frac{4km}{\sqrt{l^2 + 4a^2}} \left[ \left\{1 - \frac{4y(l-y)}{l^2 + 4a^2}\right\}^{-\frac{1}{2}} - \left\{1 + \frac{4y(l+y)}{l^2 + 4a^2}\right\}^{-\frac{1}{2}} \right] \\
 &= -\frac{4km}{\sqrt{l^2 + 4a^2}} \left[ \left\{1 + \frac{2ly}{l^2 + 4a^2} + \dots\right\} - \left\{1 - \frac{2ly}{l^2 + 4a^2} + \dots\right\} \right] \\
 &= -\frac{16kml}{(l^2 + 4a^2)^{\frac{3}{2}}} y, \quad \text{neglecting higher powers of } y.
 \end{aligned}$$

The motion is therefore Simple Harmonic with period

$$\frac{\pi}{2} \sqrt{\frac{(l^2 + 4a^2)^{\frac{3}{2}}}{kml}}.$$

### 39. Attraction of a Thin Circular Disc.

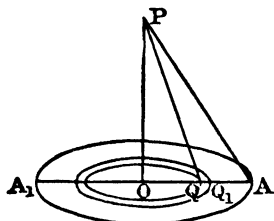


Fig. 66

P is the position of a particle of mass  $M$  and  $P$  lies on the perpendicular through the centre  $O$  of the disc at distance  $d$  from it.  $AA_1$  is a diameter of the disc whose radius is  $a$  and whose mass per unit area is  $m$ .

Consider an element in the form of an annulus and let  $OQ = x$ ,  $QQ_1 = \delta x$ ,  $\widehat{OPQ} = \theta$ ,  $\widehat{OPA} = \alpha$ .

The resultant attraction of the element on the particle is, by symmetry, along  $PO$  and its magnitude is

$$\frac{kM \cdot 2\pi x \delta x m}{PQ^2} \cdot \cos \theta.$$

Now  $x = d \tan \theta$ ,  $\delta x = d \sec^2 \theta \cdot \delta \theta$  and  $PQ = d \sec \theta$ .

Hence the resultant attraction of the disc.

$$= 2k\pi Mm \int_0^\alpha \sin \theta \, d\theta = 2k\pi Mm(1 - \cos \alpha).$$

**Ex. XV.** Find the attraction of a solid right circular cylinder, radius  $a$ , height  $h$  and density  $\rho$  on a particle of unit mass placed on the axis produced at a distance  $p$  from the nearer circular face. Hence find the ratio of the radius of the base to the height of a right circular cylinder of given volume  $V$ , so that the attraction at the centre of one of the circular ends may be a maximum.

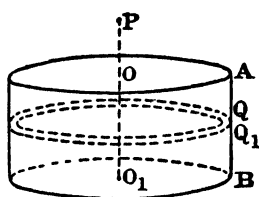


Fig. 67.

$PO = p$ ,  $OA = a$ ,  $AB = h$ .

Consider an element  $QQ_1$  in the form of a slice between two planes perp. to the axis; let  $AQ = x$ ,  $QQ_1 = \delta x$ ,  $PQ = r$ .

Attraction of element

$$= 2k\pi\rho\delta x \left(1 - \frac{p+x}{r}\right).$$

$$\text{Now } r^2 = (p+x)^2 + a^2$$

$$\therefore r\delta r = (p+x)dx,$$

and therefore attraction of cylinder

$$= 2k\pi\rho \left[ \int_0^h dx - \int_{\sqrt{a^2+p^2}}^{\sqrt{a^2+(p+h)^2}} dr \right]$$

$$= 2k\pi\rho \left[ h + \sqrt{a^2+p^2} - \sqrt{a^2+(p+h)^2} \right].$$

Let  $A$  be the attraction of the cylinder volume  $V$ , then  $p = 0$  and

$$h = \frac{V}{\pi a^2}.$$

$$\therefore A = 2k\pi\rho \left[ \frac{V}{\pi a^2} + a - \sqrt{a^2 + \frac{V^2}{\pi^2 a^4}} \right].$$

$$\therefore \frac{dA}{da} = 2k\pi\rho \left[ -\frac{2V}{\pi a^3} + 1 - \frac{2a - \frac{4V^2}{\pi^2 a^5}}{2\sqrt{a^2 + \frac{V^2}{\pi^2 a^4}}} \right].$$

$\therefore A$  is a maximum or a minimum when

$$4\left(a^2 + \frac{V^2}{\pi^2 a^4}\right)\left(1 - \frac{2V}{\pi a^3}\right)^2 = \left(2a - \frac{4V^2}{\pi^2 a^5}\right)^2,$$

$$\text{i.e. when } 4\left(\frac{V}{\pi a^3}\right)^2 - 9\left(\frac{V}{\pi a^3}\right) + 4 = 0,$$

$$\text{i.e. when } \frac{V}{\pi a^3} = \frac{9 \pm \sqrt{17}}{8}.$$

Now

$$\frac{a}{h} = \frac{\pi a^3}{\pi a^2 h} = \frac{\pi a^3}{V}$$

$$\therefore A \text{ is a max. or a min. when } \frac{a}{h} = \frac{8}{9 \pm \sqrt{17}} = \frac{9 \mp \sqrt{17}}{8}.$$

When  $\frac{a}{h} = \frac{9 - \sqrt{17}}{8}$ ,  $a^3 = \frac{9 - \sqrt{17}}{8} \cdot \frac{V}{\pi}$  and this value of  $a$  makes

$\frac{d^2 A}{da^2}$  negative.

When therefore the attraction is a maximum, the ratio is

$$\frac{1}{8}(9 - \sqrt{17}).$$

**NOTE.** The Potential at any point P is the work done by the attracting forces on a particle of unit mass as it moves from an infinite distance along any path (attraction is a conservative force) to the point P. The potential

of a particle, mass M, at distance  $x$  from it is  $-kM \int_{\infty}^x \frac{dx}{x^2}$  (—sign because  $\delta x$

is negative)  $= \frac{kM}{x}$ . Thus in finding the potential at a point due to the attraction of a body, either the force of attraction can be found first and the result integrated or  $\frac{kM}{x}$  can be summed for every element composing the body.

If we take, for example, a disc (see figure 66), the potential at distance  $y$  is  $-2k\pi m \int_{\infty}^y \left(1 - \frac{y}{\sqrt{a^2 + y^2}}\right) dx = 2k\pi m (\sqrt{a^2 + y^2} - y)$ .

By the other method, the potential at distance  $y$  of the annulus is  $2k\pi m x \delta x / PQ$ .

Now if  $PQ = r$  and  $y$  remains constant  $r^2 = x^2 + y^2 \therefore r \delta r = x \delta x$

$\therefore$  the potential of the whole disc  $= 2k\pi m \int_y^{\sqrt{a^2 + y^2}} \frac{r \delta r}{r} = 2k\pi m [\sqrt{a^2 + y^2} - y]$

Again, the attraction at a point could be found by first finding the potential at a point distance  $y$ , differentiating the result and changing sign. Thus attraction of disc at distance  $y$

$$= -2k\pi m \frac{d}{dy} [\sqrt{a^2 + y^2} - y] = 2k\pi m \left[1 - \frac{y}{\sqrt{a^2 + y^2}}\right].$$

#### 40. Attraction of a Thin Spherical Shell.

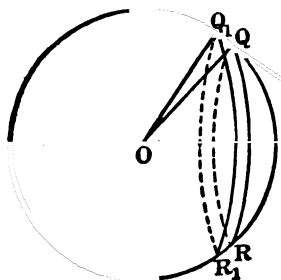


Fig. 68.

P is the position of a particle of mass M and O is the centre of a thin spherical shell radius  $a$  and mass  $m$  per unit area,  $OP = d$ . Consider an element  $QQ_1R_1R$  in the form of an annulus whose axis is  $OP$ .

Let  $PQ = y$ ,  $\widehat{POQ} = \theta$  and  $\widehat{QOQ_1} = \delta\theta$ .

The resultant attraction of every such element acts along  $PO$ .

$$\text{Res. attn. of element} = \frac{kM \cdot 2\pi a \sin \theta \cdot a \delta\theta m}{y^2} \cdot \frac{d - a \cos \theta}{y}.$$

$$\text{Now} \quad y^2 = a^2 + d^2 - 2ad \cos \theta$$

$$\therefore y \delta y = ad \sin \theta \cdot \delta\theta$$

$$\text{and} \quad 2d(d - a \cos \theta) = y^2 + d^2 - a^2.$$

$$\therefore \text{res. attn. of element} = \frac{\pi k M m a}{d^3} \cdot \frac{(y^2 + d^2 - a^2) \delta y}{y^3}$$

$$\begin{aligned} \therefore \text{res. attn. of spherical shell} &= \frac{\pi k M m a}{d^3} \int_{d-a}^{d+a} \left[ \frac{1}{y} + \frac{d^2 - a^2}{y^3} \right] dy \\ &= \frac{\pi k M m a}{d^3} \left[ (d+a) - (d-a) - (d-a) + (d+a) \right] \\ &= \frac{k M \cdot 4 \pi a^2 m}{d^2} \\ &= \frac{k M M_1}{d^2}, \text{ where } M_1 \text{ is the whole mass of the shell.} \end{aligned}$$

Thus the resultant attraction is the same as if the whole mass of the spherical shell were concentrated at the centre.

If P is inside the spherical shell,  $d < a$  and

$$\begin{aligned} \therefore \text{resultant attraction} &= \frac{\pi k M m a}{d^2} \int_{a-d}^{a+d} \left[ \frac{1}{y} - \frac{a^2 - d^2}{y^3} \right] dy \\ &= \frac{\pi k M m a}{d^2} \left[ (a+d) + (a-d) - (a-d) - (a+d) \right] = 0. \end{aligned}$$

Thus a particle anywhere inside a spherical shell is in equilibrium under its attraction.

**41. Attraction of a Solid Sphere.**—Let  $\rho$  be the density of the sphere, and suppose the sphere to be divided into thin concentric spherical shells. Also suppose  $d > a$ .

Referring to Fig. 68 we see that the attraction of an element, radius  $x$  and thickness  $\delta x$  is  $\frac{k M \cdot 4 \pi x^2 \rho \delta x}{d^2}$ .

$$\begin{aligned} \text{Resultant attraction of the solid sphere} &= \frac{4 \pi k \rho M}{d^2} \int_0^a x^2 dx \\ &= \frac{k M \cdot \frac{4}{3} \pi a^3 \rho}{d^2} \\ &= \frac{k M M_1}{d^2}, \end{aligned}$$

where  $M_1$  is the whole mass of the sphere.

**Ex. XVI.** The attraction of the earth on a particle of unit mass on the earth's surface being  $g$  poundals, find the attraction at a distance  $h$  feet below the surface. Consider the earth as having uniform density and radius  $a$  feet.

The resultant attraction on the particle, of the hollow sphere

having radii  $(a - h)$  ft. and  $a$  ft., is zero and hence the resultant attraction of the whole earth is that of a sphere of radius  $(a - h)$  ft. acting on the particle on its surface.

Therefore the resultant attraction of the earth on the particle at depth  $h$  ft.

$$= \frac{k \frac{4}{3} \pi (a - h)^3 \cdot \rho}{(a - h)^2} \text{ pdl.}$$

$$= k \cdot \frac{4}{3} \pi \rho (a - h). \text{ pdl.}$$

Now considering the particle of unit mass on the earth's surface, we have

$$g = \frac{k \cdot \frac{4}{3} \pi a^3 \rho}{a^2} = k \frac{4}{3} \pi \rho a.$$

Therefore the resultant attraction of the earth on the particle at depth  $h$  ft.  $= g \cdot \frac{a - h}{a}$  pdl.

### Problems V.

1. Find the position of the C.G. of the lamina bounded by the curve  $y^2 = 4ax$ , the axis of  $x$  and the line  $x = a$ .

2. Find the position of the C.G. of the solid formed by rotating the curve  $y^2 = 4ax$  about the axis of  $x$  and bounded by the planes  $x = 0$  and  $x = a$ .

3. Find the C.G. of the portion of the sphere, radius  $a$  between parallel planes at distances  $b$  and  $c$  from the centre.

4. Find the centroid of the arc of the catenary  $s = c \tan \psi$  terminated by the points determined by  $\psi = 0$  and  $\psi = \frac{\pi}{4}$ .

5. Find the centroid of a quadrant of an ellipse whose axes are  $2a$  and  $2b$ . If the ellipse rotate about the major axes ( $2a$ ) find the C.G. of half the spheroid bounded by a plane containing the minor axis.

6. A curve is determined by

|     |     |     |      |     |     |     |      |      |      |      |     |      |     |     |
|-----|-----|-----|------|-----|-----|-----|------|------|------|------|-----|------|-----|-----|
| $x$ | 1.1 | 1.8 | 2.6  | 3.8 | 4.4 | 5.2 | 5.35 | 5.1  | 4.3  | 3.1  | 2.0 | 0.7  | 0.6 | 1.1 |
| $y$ | 1.0 | 0.3 | 0.08 | 0.4 | 0.7 | 1.7 | 2.6  | 3.75 | 5.17 | 5.83 | 5.5 | 3.25 | 2.1 | 1.0 |

Show graphically that the centroid of the area enclosed by the curve is approximately  $\bar{x} = 2.95$ ,  $\bar{y} = 2.88$ .

7. Find the C.G. of the larger portion of a sphere (radius  $10''$ ) whose bounding sections, which are parallel, have radii  $6''$  and  $8''$  respectively.

8. A segment, height  $h$  is cut from a sphere of radius  $a$ . Prove that the distance of the C.G. from the base is  $\frac{h(4a - h)}{4(3a - h)}$ .

9. A segment is cut from a circle of radius  $a$  and  $2a$  is the angle

## CHAPTER VI

### FRICTION : VIRTUAL WORK : STABLE AND UNSTABLE EQUILIBRIUM

#### 42. Problems on Rough Surfaces.

**Ex. I.** Find the least force which will prevent a body from sliding down a rough inclined plane whose inclination  $\alpha$  to the horizon is greater than  $\lambda$ , the angle of friction.

Suppose the body (mass  $m$ ) to be in limiting equilibrium under a force  $P$  inclined at an angle  $\theta$  to the plane. Let  $R$  be the normal reaction.

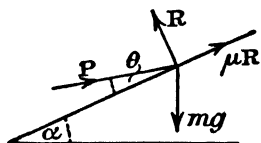


Fig. 69.

$$\text{Then } P \cos \theta = mg \sin \alpha - \mu R$$

$$\text{where } \mu = \tan \lambda.$$

$$\text{and } P \sin \theta = -mg \cos \alpha + R$$

$$\therefore P = \frac{mg(\sin \alpha - \mu \cos \alpha)}{\cos \theta + \mu \sin \theta}.$$

$P$  is a minimum when  $\cos \theta + \mu \sin \theta$  is a maximum

$$\text{i.e. when } -\sin \theta + \mu \cos \theta = 0$$

$$\text{i.e. when } \tan \theta = \mu \text{ or } \theta = \lambda.$$

$$\text{i.e. when } \cos \theta + \mu \sin \theta = \sqrt{1 + \mu^2} = \sec \lambda.$$

$$\therefore \text{the least force is } mg \cos \lambda (\sin \alpha - \tan \lambda \cos \alpha) = mg \sin(\alpha - \lambda).$$

**Ex. II.** A uniform rod, length  $2l$ , weight  $W$  rests on a rough horizontal surface whose co-efficient of friction is  $\mu$ . It is pulled perpendicular to its length by a string attached to a point distant  $a$  from the centre. Find the point about which it would commence to turn and the least force required to move the rod. What would be the value of  $\mu$  if a force of  $\frac{W}{3}$  applied at one end just moves it?

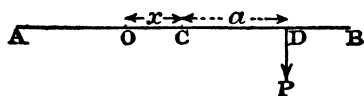


Fig. 70.

AB represents the rod, C is the middle point, D is the point of application of the force  $P$  which causes the rod to be on the point of rotating about a point O, where  $OC = x$ .

The force of friction is  $\frac{\mu W}{2l}$  per unit length and acts in the opposite direction from  $P$  at every point of the portion OB and in the same direction at every point of the portion AO;

$$\therefore P = \frac{\mu W}{2l} \left[ (l + x) - (l - x) \right] = \frac{\mu W x}{l}. \quad \dots \quad (1)$$

Taking moments about O we have,

$$P(a + x) = \frac{\mu W}{2l} \left[ \frac{(l + x)^2}{2} + \frac{(l - x)^2}{2} \right] = \frac{\mu W}{2l} (l^2 + x^2) \quad (2)$$

$$\therefore \frac{\mu W x(a+x)}{l} = \frac{\mu W}{2l} (l^2 + x^2)$$

$$\therefore x^2 + 2ax - l^2 = 0$$

$$\therefore x = -a \pm \sqrt{l^2 + a^2}.$$

The plus sign must be taken for if  $x$  be negative and numerically greater than  $a$ , the L.H.S. of equation (2) is negative whereas the R.H.S. is positive.

Thus the rod is just on the point of moving about a point on the side of C remote from D and at a distance  $\sqrt{l^2 + a^2} - a$  from C.

Also the rod will move if the force exceed  $\frac{\mu W}{l} [\sqrt{l^2 + a^2} - a]$ .

When  $P = \frac{W}{3}$  and  $a = l$ , we have  $\frac{W}{3} = \frac{\mu W}{l} (\sqrt{2} - 1)l$

so that 
$$\mu = \frac{\sqrt{2} + 1}{3}.$$

**Ex. III.** A uniform circular plate lies on a rough inclined plane whose co-efficient of friction is  $\mu$  and which is inclined at an angle  $\alpha$  to the horizontal. If it be free to move about a fixed point on its circumference, prove that it will rest in all positions if  $\tan \alpha \geq \frac{32\mu}{9\pi}$ .

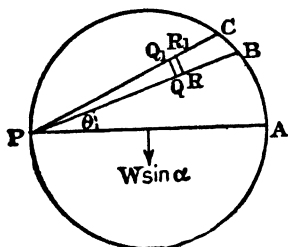


Fig. 71.

P is the fixed point. Let the weight of the plate be  $W$  and the radius  $a$ . The moment of the weight of the plate about P will be greatest when the diameter PA is horizontal so that if the plate be in limiting equilibrium in this position, it will rest in all positions on the plane.

The normal reaction is  $W \cos \alpha$  and the force of friction is  $\frac{\mu W \cos \alpha}{\pi a^2}$  per unit area.

Let  $\widehat{APQ} = \theta$ ,  $\widehat{PQ_1} = \delta\theta$ ,  $PQ = r$  and  $QR = \delta r$ , then moment of friction acting on  $QRR_1Q_1$  about P

$$= \frac{\mu W \cos \alpha}{\pi a^2} r^2 \delta r \delta \theta.$$

$\therefore$  moment of friction acting on PBC about P

$$\begin{aligned} &= \frac{\mu W \cos \alpha \cdot \delta \theta}{\pi a^2} \int_0^{2a \cos \theta} r^2 dr \\ &= \frac{8\mu W a \cos \alpha \cos^3 \theta \delta \theta}{3\pi}. \end{aligned}$$

$\therefore$  moment of friction acting on the whole plate about P

$$\begin{aligned} &= \frac{16\mu W a \cos \alpha}{3\pi} \int_0^{\frac{\pi}{2}} (\cos 3\theta + 3 \cos \theta) d\theta. \\ &= \frac{32 \mu W a \cos \alpha}{9\pi}. \end{aligned}$$

The plate will therefore be in equilibrium in all positions if

$$W \sin \alpha \cdot a \succ \frac{32\mu Wa \cos \alpha}{9\pi}$$

i.e. if  $\tan \alpha \succ \frac{32\mu}{9\pi}$ .

### 43. Equilibrium of a Particle on a Rough Curve in One Plane.

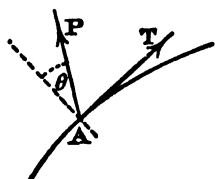


Fig. 72

A is the position of the particle on the curve. Suppose P is the resultant of the external forces (reaction of the wire and force of friction of the wire excluded) and that its direction makes an angle  $\theta$  with the normal to the curve at A. Let T be the sum of the components of the external forces along the tangent to the curve at A.

If  $\tan \theta \succ \mu$ , the co-efficient of friction, equilibrium will be maintained, but  $\tan^2 \theta = \frac{T^2}{P^2 - T^2}$  so that if  $\frac{T^2}{P^2 - T^2} \succ \mu^2$  there will be equilibrium.

In the case of a particle on a curve in a vertical plane, the only external force being its weight, the position of limiting equilibrium will be at that point on the curve where the tangent is inclined to the horizontal at an angle  $\lambda$  where  $\tan \lambda = \mu$ .

**Ex. IV.** A heavy particle rests in limiting equilibrium on a rough cycloid, axis vertical and vertex downwards. Show that the altitude of the particle above the vertex is  $\frac{2a\mu^2}{1 + \mu^2}$ , where  $a$  is the radius of the generating circle and  $\mu$  is the co-efficient of friction.

The ordinate of any point P on the cycloid is given by  $y = a(1 - \cos \theta)$ , where  $\frac{\theta}{2}$  is the angle the tangent at P makes with Ox, the tangent at the vertex O.

If P be the position of limiting equilibrium of the particle, then  $\mu = \tan \frac{\theta}{2}$ .

$$\text{Now } y = 2a \sin^2 \frac{\theta}{2} = 2a \frac{\tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \text{ and thus the height of the}$$

particle above the vertex when in limiting equilibrium is  $\frac{2a\mu^2}{1 + \mu^2}$ .



**Ex. V.** A particle is on a rough wire ( $\mu = \frac{1}{2}$ ) in the shape of an ellipse  $x^2 + 2y^2 = 1$ . It is under the action of external forces  $X$  and  $Y$  respectively parallel to the axes. Find positions of limiting equilibrium.

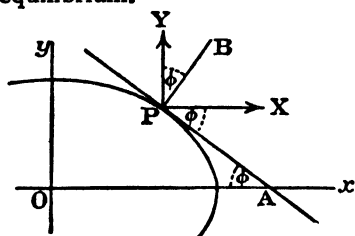


Fig. 73.

Let  $P(x_1, y_1)$  be a position of limiting equilibrium.  $PA$  is the tangent and  $PB$  the normal at  $P$ .

From the equation of the curve,

$$\frac{dy}{dx} = -\frac{x}{2y}$$

so that if

$$\widehat{OAP} = \phi, \quad \tan \phi = \frac{x_1}{2y_1}$$

and therefore  $\sin \phi = \frac{x_1}{\sqrt{x_1^2 + 4y_1^2}}$  and  $\cos \phi = \frac{2y_1}{\sqrt{x_1^2 + 4y_1^2}}$ .

Sum of the components along  $PA = X \cos \phi - Y \sin \phi$

Sum of the components along  $PB = X \sin \phi + Y \cos \phi$ .

If  $\theta$  be the angle the direction of the resultant of  $X$  and  $Y$  makes with  $PB$ , then

$$\tan \theta = \frac{X \cos \phi - Y \sin \phi}{X \sin \phi + Y \cos \phi} = \frac{2y_1 X - x_1 Y}{x_1 X + 2y_1 Y}.$$

For the positions of limiting equilibrium,

$$\frac{2y_1 X - x_1 Y}{x_1 X + 2y_1 Y} = \pm \frac{1}{2}.$$

Taking the +ve sign,  $y_1 = \frac{(x_1 X + 2Y)}{2(2X - Y)}$ , from which

$$x_1 = \frac{\pm \sqrt{2}(2X - Y)}{\sqrt{9X^2 - 4XY + 6Y^2}}, \quad y_1 = \frac{\pm (X + 2Y)}{\sqrt{2}(9X^2 - 4XY + 6Y^2)}.$$

Taking the -ve sign,  $y_1 = \frac{x_1(2Y - X)}{2(2X + Y)}$  from which

$$x_1 = \frac{\pm \sqrt{2}(2X + Y)}{\sqrt{9X^2 + 4XY + 6Y^2}}, \quad y_1 = \frac{\pm (2Y - X)}{\sqrt{2}(9X^2 + 4XY + 6Y^2)}.$$

**NOTE 1.**—If  $X \cos \phi < Y \sin \phi$ ,  $\tan \theta$  will be negative, hence the minus sign when equating  $\tan \theta$  to  $\mu$ . The particle would be in equilibrium in the first quadrant ( $2X - Y$  and  $2Y - X$  being +ve) between the points

$$\frac{\sqrt{2}(2X - Y)}{\sqrt{9X^2 - 4XY + 6Y^2}}, \quad \frac{X + 2Y}{\sqrt{2}(9X^2 - 4XY + 6Y^2)}$$

and

$$\frac{\sqrt{2}(2X + Y)}{\sqrt{9X^2 + 4XY + 6Y^2}}, \quad \frac{(2Y - X)}{\sqrt{2}(9X^2 + 4XY + 6Y^2)}.$$

**44. Virtual Work.** Consider a system of particles acted upon by a system of forces in equilibrium. Imagine a slight displacement of the material system consistent with the geometrical conditions determining the configuration. If

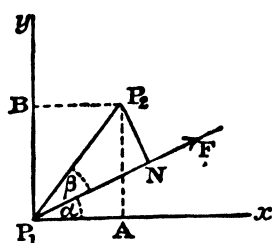


Fig. 74.

Taking  $P_1$  as origin, let  $\widehat{NP_1x} = \alpha$ ,  $\widehat{P_2P_1N} = \beta$  and let  $X, Y$  be the components of  $F$  parallel to  $P_1x, P_1y$  respectively. Then the sum of the virtual works of  $X$  and  $Y$  is

$$\begin{aligned} & X \cdot P_1P_2 \cos(\alpha + \beta) + Y \cdot P_1P_2 \sin(\alpha + \beta) \\ &= F \cdot P_1P_2 [\cos \alpha \cdot \cos(\alpha + \beta) + \sin \alpha \cdot \sin(\alpha + \beta)] \\ &= F \cdot P_1P_2 \cos \beta \\ &= F \cdot P_1N. \end{aligned}$$

Thus the virtual work of a force is equal to the sum of the virtual works of its components.

The Principle of Virtual Work states that—If a system of forces acting on a system of particles be in equilibrium and the latter system undergoes a displacement consistent with the geometrical conditions determining the configuration and involving small quantities of the first order of magnitude, then the algebraic sum of the virtual works done by the forces is a small quantity of at least the second order of magnitude.

The proof of this principle in the case of forces in one plane is given as follows :—

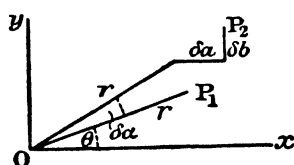


Fig. 75.

$P_1$  is the point of application of a force  $F$  before displacement. The most general displacement of a body can be made by rotating it about a point and then applying a motion of translation. Let the body be rotated through an angle  $\delta\alpha$  about  $O$  and then translated distances  $\delta a, \delta b$  parallel to the fixed axes  $Ox, Oy$ . Let  $P_2$  be the point of application of  $F$  after displacement and suppose  $\delta a, \delta\alpha, \delta b$  are small quantities of the first order of magnitude. Let  $r, \theta$  be the polar co-ordinates of  $P_1$  and  $x, y$ ;  $(x + \delta x), (y + \delta y)$  the cartesian co-ordinates of  $P_1, P_2$  respectively. Then

$$\begin{aligned} \delta x &= r \cos(\theta + \delta\alpha) + \delta a - r \cos \theta \\ &= r \left[ \cos \theta \cdot \left( 1 - \frac{\delta\alpha^2}{2} + \dots \right) - \sin \theta \cdot \left( \delta\alpha - \frac{\delta\alpha^3}{6} + \dots \right) \right] \\ &\quad + \delta a - r \cos \theta \end{aligned}$$

$$\begin{aligned}
 &= \delta a - r \sin \theta \cdot \delta a - r \cos \theta \cdot \frac{\delta a^2}{2} + \text{small quantities of higher order} \\
 &= \delta a - y \delta a - \frac{1}{2} x \delta a^2 + \text{,, ,, ,,} \\
 \text{Similarly } \delta y &= \delta b + x \delta a - \frac{1}{2} y \delta a^2 + \text{,, ,, ,,}
 \end{aligned}$$

If  $X, Y$  be the components of  $F$  parallel to  $Ox, Oy$ , then the virtual work of  $F$  is

$$\begin{aligned}
 &X \cdot \delta x + Y \cdot \delta y \\
 &= X \cdot \delta a + Y \delta b + (Yx - Xy) \cdot \delta a - \frac{\delta a^2}{2} (Xx + Yy) + \dots \dots \dots \\
 &\text{small quantities of higher order.}
 \end{aligned}$$

Now  $\delta a, \delta a, \delta b$  are the same for the displaced points of application of the other forces of the system and therefore the virtual work of the system is

$$\begin{aligned}
 &\delta a \cdot \Sigma X + \delta b \cdot \Sigma Y + \delta a \cdot \Sigma (Yx - Xy) - \frac{\delta a^2}{2} \Sigma (Xx + Yy) \\
 &\quad + \dots \text{small quantities of higher order.}
 \end{aligned}$$

In the position from which the body was displaced, the system of forces is in equilibrium and therefore  $\Sigma X = 0$ ,  $\Sigma Y = 0$  and  $\Sigma (Yx - Xy) = 0$  and thus the virtual work of the system is

$$-\frac{\delta a^2}{2} \Sigma (Xx + Yy) + \text{small quantities of higher order,}$$

which is a small quantity of at least the second order.

Let  $P$  denote the potential energy of the system in any position of equilibrium, then when a virtual displacement takes place the virtual work of the system of forces will be denoted by  $\delta P$ . Suppose that  $P$  is dependent on a single independent variable, say  $\phi$ , and for a virtual displacement  $\phi$  becomes  $\phi + \delta\phi$  where  $\delta\phi$  is a small quantity

of the first order, then  $\frac{\delta P}{\delta \phi}$  will be a small quantity of at least the

first order and consequently  $\lim_{\delta \phi \rightarrow 0} \frac{\delta P}{\delta \phi}$  i.e.  $\frac{dP}{d\phi}$  will be zero.

Conversely, if  $\frac{dP}{d\phi} = 0$ , the configuration thus determined is one where the system of forces is in equilibrium. For  $\delta P$  will be a small quantity of at least the second order and thus  $\delta a \Sigma X + \delta b \Sigma Y + \delta a \Sigma (Yx - Xy) = 0$ . Considering the displacement  $\delta a$  only,  $\delta b$  and  $\delta a$  vanish and thus  $\Sigma X = 0$ ; similarly for a displacement  $\delta b$  only,  $\Sigma Y = 0$  and for a displacement of simple rotation  $\delta a$  about  $O$ ,  $\Sigma (Yx - Xy) = 0$ . That is, the sum of the components of the forces parallel to each axis is zero and the sum of the moments about  $O$  vanishes and thus the system of forces is in equilibrium.

NOTE 2. Suppose  $P$  depends on independent variables  $\phi, \psi, z, \dots$ , and that  $\delta P = f_1 \delta \phi + f_2 \delta \psi + f_3 \delta z + \dots$  where  $f_1, f_2, f_3$  are functions of  $\phi, \psi, z, \dots$ . Considering a displacement where  $\phi$  changes to  $\phi + \delta \phi$ , but  $\psi, z, \dots$  remain unaltered, we have  $\delta P = f_1 \delta \phi$ ,  $\delta P$  being a small quantity

of at least the second order.  $\frac{\partial P}{\partial \phi}$  will be a small quantity of at least the first order and for positions of equilibrium  $\frac{\partial P}{\partial \phi} = 0$ . Similarly  $\frac{\partial P}{\partial \psi} = 0$  and  $\frac{\partial P}{\partial z} = 0$ , so that there are as many equations as independent variables.

**Ex. VI.** Two light rods AOC, BOD are smoothly hinged at O, a point at a distance  $c$  from each of the ends A, B which are connected by a string of length  $2c \sin \alpha$ . The rods rest in a vertical plane with the ends A, B on a smooth horizontal table. A smooth circular disc of radius  $a$  and weight  $w$  is placed on the rods above O with its plane vertical so that the rods are tangents to the disc.

Prove that the tension of the string is  $\frac{1}{2}w \left( \frac{a}{c} \operatorname{cosec}^2 \alpha + \tan \alpha \right)$ .

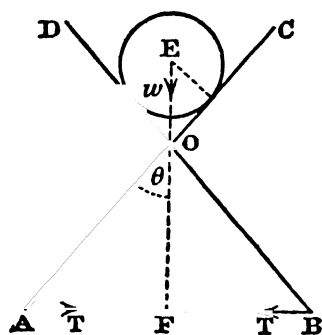


Fig. 76.

EOF is the vertical through O and E is the centre of the disc. Let  $T$  be the tension of the string and  $P$  the potential energy of the system, then

$$P = T \cdot AB + w \cdot EF + \text{a constant.}$$

Let  $\widehat{AOF} = \theta$ , then  $AB = 2c \sin \theta$  and  $EF = a \operatorname{cosec} \theta + c \cos \theta$ .

$$\therefore P = T \cdot 2c \sin \theta + w (a \operatorname{cosec} \theta + c \cos \theta) + \text{a constant.}$$

$$\therefore \frac{dP}{d\theta} = T \cdot 2c \cos \theta + w \left( -\frac{a \cos \theta}{\sin^2 \theta} - c \sin \theta \right).$$

In the position of equilibrium,  $\theta = \alpha$  and  $\frac{dP}{d\theta} = 0$

$$\therefore T = \frac{w}{2} \left( \frac{a}{c} \operatorname{cosec}^2 \alpha + \tan \alpha \right).$$

NOTE 3. If  $x$  is the height of the centre of gravity of a body of weight  $w$  above a fixed horizontal level, the potential energy is a constant  $+wx$ . When at a depth  $y$  below this level, the potential energy is a constant  $-wy$ .

If  $T$  is the tension of a string AB in a position where the system is in equilibrium and the displacement causes the length to be slightly increased to AC, then the potential energy is a constant  $+T \cdot BC$ , i.e. a constant  $+T \cdot AC - T \cdot AB$ , i.e. a constant  $+T \cdot AC$ . In the case of a rod subjected to a thrust  $T$ , the potential energy in the displaced position AC (slightly less than AB) is a constant  $-T \cdot AC$ .

**Ex. VII.** AB and AC are uniform rods of weight  $w$  per unit length and their lengths are 6 and 8 units respectively. They are freely hinged at A and the ends B and C rest on a smooth horizontal

table. Their middle points E and F are connected by a string 5 units long. Find the tension of the string.

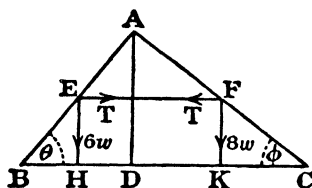


Fig. 77.

EH and FK are the perpendiculars to the table. Let T be the tension of the string and P the potential energy of the system, then

$$P = T \cdot EF + 14w \cdot EH + C, \text{ a constant.}$$

Let  $\widehat{ABC} = \theta$  and  $\widehat{ACB} = \phi$ , then

$$EF = 3 \cos \theta + 4 \cos \phi \quad \text{and} \quad EH = 3 \sin \theta$$

$$\therefore P = T \cdot (3 \cos \theta + 4 \cos \phi) + 42w \cdot \sin \theta + C$$

$$\therefore \frac{dP}{d\theta} = -3T \sin \theta - 4T \sin \phi \cdot \frac{d\phi}{d\theta} + 42w \cos \theta.$$

$$\text{Now } 3 \sin \theta = 4 \sin \phi \quad \therefore 3 \cos \theta = 4 \cos \phi \cdot \frac{d\phi}{d\theta}.$$

$$\therefore \frac{dP}{d\theta} = -3T \sin \theta - 3T \tan \phi \cdot \cos \theta + 42w \cos \theta.$$

When  $\sin \theta = \frac{4}{5}$ ,  $\cos \theta = \frac{3}{5}$ ,  $\tan \phi = \frac{3}{4}$  and the system is in equilibrium.

$$\therefore T \cdot \left( \frac{12}{5} + \frac{27}{20} \right) = \frac{126}{5} w.$$

$$\therefore T = \frac{168}{25} w.$$

**Ex. VIII.** Three equal uniform rods, weight W are freely hinged at D whose height is h and the other ends A, B, C rest on a smooth horizontal plane. A weight  $W_1$  is supported at D and strings each of length l join A, B, C to each of the middle points of the other two rods. Find the tension of the strings.

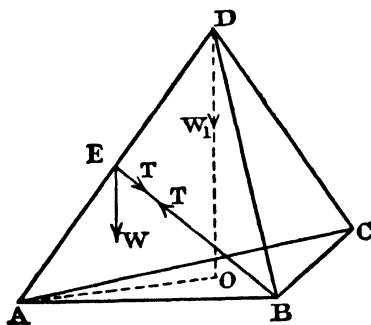


Fig. 78.

DO is the vertical through D and BE is one of the six strings. Let T be the tension of each string and P the potential energy of the system, then

$$P = 6T \cdot BE + \left( W_1 + \frac{3W}{2} \right) \cdot DO + C \text{ (a constant), } \dots \dots (1)$$

Let  $DO = x$ ,  $BE = y$ ,  $AD = L$  (a constant),  $AB = a$  (a variable),

$$\text{then as } AO^2 = \frac{a^2}{3}, \text{ we have } L^2 = x^2 + \frac{a^2}{3}.$$

$$\text{Also from the } \triangle ABD, \quad 2y^2 + \frac{L^2}{2} = a^2 + L^2.$$

Eliminating  $a$ , we have  $2y^2 = \frac{L^2}{2} + 3L^2 - 3x^2$

$$\therefore 2y \frac{dy}{dx} = -3x$$

Eqn. (1) becomes  $P = 6Ty + \left(W_1 + \frac{3W}{2}\right)x + C$ ,

$$\begin{aligned} \therefore \frac{dP}{dx} &= 6T \frac{dy}{dx} + \left(W_1 + \frac{3W}{2}\right) \\ &= -\frac{9Tx}{y} + \left(W_1 + \frac{3W}{2}\right) \dots \dots (2) \end{aligned}$$

In the position of equilibrium,  $x = h$ ,  $y = l$  and  $\frac{dP}{dx} = 0$ ; therefore equating the right hand side of eqn. (2) to zero, we have

$$T = \frac{l(2W_1 + 3W)}{18h}$$

**45. Stable and Unstable Equilibrium.** Consider a system of particles such that the configuration and consequently the potential  $P$  of the forces acting are completely determined by one independent variable, say  $x$ , then the equation  $\frac{dP}{dx} = 0$  determines the positions of equilibrium when all the forces of the system are known.

As the function  $P$  is a maximum or minimum for values of  $x$  given by the equation  $\frac{dP}{dx} = 0$ , it follows that in positions of equilibrium, the potential energy of the system of forces is a maximum or minimum. In positions of stable equilibrium the potential energy of the system of forces is a minimum, for if a displacement be made, work will be done against the forces whose potential energy will thereby be increased. Similarly, in positions of unstable equilibrium, the potential energy of the system of forces will be a maximum. Consequently a position of equilibrium is stable or unstable according as  $\frac{d^2P}{dx^2}$  is positive or negative. If  $P$  is constant, the system will be in neutral equilibrium.

In particular, if during changes of configuration of the system, gravity is the only force which does work, the potential energy in any position will be  $W\bar{y} + \text{Const.}$  where  $W$  is the total weight of the particles and  $\bar{y}$  the distance of the centre of gravity above some fixed horizontal line. As  $W$

is constant, the positions of equilibrium are determined by the equation  $d\bar{y}/dx=0$  and the equilibrium is stable or unstable according as  $d^2\bar{y}/dx^2$  is positive or negative.

NOTE 4. As the system is displaced, the centre of gravity will describe a curve. At positions where the tangent to this curve is horizontal, there will be equilibrium. If displacement from a minimum point take place, the C.G. rises and consequently gravity tends to restore the equilibrium. Displaced from a maximum point, the C.G. falls and gravity tends to move the system from this position of equilibrium. If  $\frac{d^2P}{dx^2} = \frac{d^3P}{dx^3} = \dots = \frac{d^{2n-1}P}{dx^{2n-1}} = 0$

and  $\frac{d^{2n}P}{dx^{2n}}$  is not zero, then the equilibrium is stable or unstable according as

$\frac{d^{2n}P}{dx^{2n}}$  is positive or negative.

**Ex. IX.** A solid circular cone whose height is  $h$  and semi-vertical angle  $\alpha$  is placed vertex downwards, in a smooth circular hole whose radius is  $a$ , cut in a horizontal table. Find the positions of equilibrium. Show that the symmetrical position of equilibrium is stable provided that  $16a > 3h \sin 2\alpha$  and that the position of equilibrium when the axis is inclined to the vertical is stable if  $16a < 3h \sin 2\alpha$ .

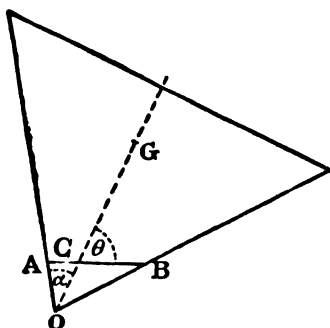


Fig. 79.

Let  $\angle GCB = \theta$ , then  $\frac{OC}{AB} = \frac{OC}{OA} \cdot \frac{OA}{AB} = \frac{\sin(\theta + \alpha)}{\sin \theta} \cdot \frac{\sin(\theta - \alpha)}{\sin 2\alpha}$ .

$$\therefore OC = \frac{2a \sin(\theta + \alpha) \sin(\theta - \alpha)}{\sin \theta \cdot \sin 2\alpha} = \frac{a(\cos 2\alpha - \cos 2\theta)}{\sin 2\alpha \cdot \sin \theta}.$$

Let  $\bar{y} \equiv$  ht. of G above AB, then

$$\bar{y} = \left[ \frac{3}{4}h - \frac{a(\cos 2\alpha - \cos 2\theta)}{\sin 2\alpha \cdot \sin \theta} \right] \sin \theta = \frac{3}{4}h \sin \theta + \frac{a \cos 2\theta}{\sin 2\alpha} - a \cot 2\alpha$$

$$\therefore \frac{d\bar{y}}{d\theta} = \cos \theta \left[ \frac{3}{4}h - \frac{4a \sin \theta}{\sin 2\alpha} \right].$$

$\therefore$  the cone is in equilibrium when  $\cos \theta = 0$  and when  $16a \sin \theta = 3h \sin 2\alpha$ .

i.e. when  $\theta = \frac{\pi}{2}$  or  $\sin^{-1} \frac{3h \sin 2\alpha}{16a}$ .

$$\text{Now } \frac{d^2\bar{y}}{d\theta^2} = -\frac{3}{4}h \sin \theta - \frac{4a \cos 2\theta}{\sin 2\alpha}.$$

When  $\theta = \frac{\pi}{2}$ ,  $\frac{d^2\bar{y}}{d\theta^2}$  is +ve if  $-\frac{3}{4}h + \frac{4a}{\sin 2\alpha} > 0$

∴ the symmetrical position of equilibrium is stable if

$$16a > 3h \sin 2a.$$

When  $\sin \theta = \frac{3h \sin 2a}{16a}$ ,  $\frac{d^2\bar{y}}{d\theta^2}$  is +ve if

$$-\frac{9h^2 \sin 2a}{64a} - \frac{4a}{\sin 2a} \left[ 1 - \frac{9h^2 \sin^2 2a}{128a^2} \right] > 0.$$

i.e. if  $\frac{9h^2 \sin 2a}{64} > \frac{4a}{\sin 2a}$ .

∴ the oblique position of equilibrium is stable if  $16a < 3h \sin 2a$ .

**Ex. X.** A uniform heavy rod hangs with two smooth rings fixed at its ends the rings sliding on two smooth wires OA, OB which are in the same vertical plane and below the point O. If the inclinations of the wires to the horizontal are  $\alpha$  and  $\beta$  ( $\beta > \alpha$ ) show that one position of equilibrium is given by  $2 \tan \theta = \cot \alpha - \cot \beta$ , where  $\theta$  is the inclination of the rod to the horizontal. Prove that this position is stable.

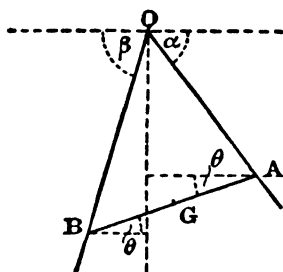


Fig. 80.

Let AB represent the rod whose length is  $l$ , and whose inclination to the horizontal is  $\theta$ . Let G be the C.G. of the rod and let  $OA = a$  and  $OB = b$ ; then if  $\bar{y}$  is the depth of G vertically below O,

$$\bar{y} = \frac{1}{2}(a \sin \alpha + b \sin \beta).$$

Now  $\frac{a}{l} = \frac{\sin(\beta - \theta)}{\sin(\alpha + \beta)}$  and  $\frac{b}{l} = \frac{\sin(\alpha + \theta)}{\sin(\alpha + \beta)}$ .

$$\therefore \bar{y} = \frac{l}{2 \sin(\alpha + \beta)} \left[ \sin \alpha \cdot \sin(\beta - \theta) + \sin \beta \cdot \sin(\alpha + \theta) \right]$$

$$\therefore \frac{d\bar{y}}{d\theta} = \frac{l}{2 \sin(\alpha + \beta)} \left[ \sin \beta \cos(\alpha + \theta) - \sin \alpha \cdot \cos(\beta - \theta) \right]$$

∴ the rod is in equilibrium when

$$\cos \theta (\sin \beta \cos \alpha - \sin \alpha \cos \beta) - 2 \sin \alpha \sin \beta \sin \theta = 0$$

i.e. when

$$2 \tan \theta = \cot \alpha - \cot \beta.$$

$$\text{Now } \frac{d^2\bar{y}}{d\theta^2} = \frac{l}{2 \sin(\alpha + \beta)} \left[ -\sin \beta \sin(\alpha + \theta) - \sin \alpha \sin(\beta - \theta) \right]$$

As  $\bar{y}$  is measured downwards, the equilibrium will be stable when  $\bar{y}$  is a maximum, that is when  $\frac{d^2\bar{y}}{d\theta^2}$  is -ve.

Now  $\alpha$ ,  $\beta$  and  $\theta$  all being angles between 0 and  $\frac{\pi}{2}$ ,  $\sin \beta \sin(\alpha + \theta)$  is +ve and as  $\beta > \theta$  (A being below O),  $\sin \alpha \sin(\beta - \theta)$  is +ve.

Thus  $\frac{d^2\bar{y}}{d\theta^2}$  is negative and the equilibrium is stable.



**Ex. XI.** A thin wire has the form of a circle in a vertical plane with centre C. A, B are pegs attached to the wire so that CA, CB make angles  $\alpha$  on opposite sides of the downward vertical through C. A small ring of weight M can slide on the wire, and is attached to two strings passed over the pegs with weights  $m$  hanging from their ends. Write down the potential energy of the system when the radius of M makes an angle  $\theta$  with the vertical.

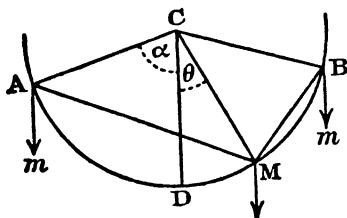


Fig. 81.

Hence discuss the stability of equilibrium positions in the cases

$$M \leq m \sin \frac{1}{2} \alpha.$$

Let  $a$  be the radius of the circle and CD the vertical radius. Let  $P_1$  be the potential energy of the system when the mass M is at D, then when the radius to M makes an angle  $\theta$  with the vertical, the potential energy  $P$  of the system

is given by

$$\begin{aligned} P &= P_1 + Ma(1 - \cos \theta) + m(AM - AD) - m(BD - BM) \\ &= P_1 + Ma(1 - \cos \theta) + 2ma \left( \sin \frac{\alpha + \theta}{2} - \sin \frac{\alpha}{2} \right) \\ &\quad - 2ma \left( \sin \frac{\alpha}{2} - \sin \frac{\alpha - \theta}{2} \right) \\ &= P_1 + Ma(1 - \cos \theta) + 4ma \sin \frac{\alpha}{2} \cos \frac{\theta}{2} - 4ma \sin \frac{\alpha}{2}. \end{aligned}$$

$$\begin{aligned} \therefore \frac{dP}{d\theta} &= Ma \sin \theta - 2ma \sin \frac{\alpha}{2} \sin \frac{\theta}{2} \\ &= 2a \sin \frac{\theta}{2} \left[ M \cos \frac{\theta}{2} - m \sin \frac{\alpha}{2} \right]. \end{aligned}$$

$\therefore$  the system is in equilibrium when  $\theta = 0$  and when  $\cos \frac{\theta}{2} = \frac{m}{M} \sin \frac{\alpha}{2}$ .

$$\text{Now } \frac{d^2P}{d\theta^2} = a \left[ M \cos \theta - m \sin \frac{\alpha}{2} \cos \frac{\theta}{2} \right].$$

$\therefore$  when  $\theta = 0$ , the equilibrium is stable or unstable according as  $M \gtrless m \sin \frac{\alpha}{2}$ .

Also, when  $\cos \frac{\theta}{2} = \frac{m}{M} \sin \frac{\alpha}{2}$  the equilibrium is stable or unstable according as  $\frac{2m^2}{M} \sin^2 \frac{\alpha}{2} - M - \frac{m^2 \sin^2 \alpha}{M} \leq 0$

i.e. according as  $M \leq m \sin \frac{\alpha}{2}$ .

**Ex. XII.** A uniform rod of length  $l$  rests horizontally with its ends in contact with a smooth parabolic wire whose axis is vertical and vertex downwards. Prove that the equilibrium is stable or unstable according as the length of the rod is less than or greater than the latus rectum of the parabola.

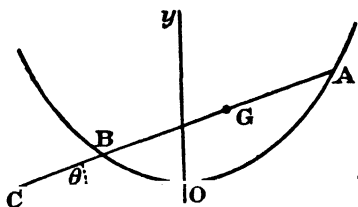


Fig. 82.

of the rod. Let the co-ordinates of A and B be  $x_1, y_1$  and  $x_2, y_2$ , and  $\bar{y}$  the height of the middle point of the rod,

$$\text{then} \quad \bar{y} = \frac{1}{2}(y_1 + y_2) = \frac{1}{8a}(x_1^2 + x_2^2),$$

$$\therefore \quad \frac{d\bar{y}}{dx_1} = \frac{1}{4a}\left(x_1 + x_2 \frac{dx_2}{dx_1}\right). \quad \dots \quad \dots \quad (1)$$

$$\text{Now} \quad x_1^2 - x_2^2 = 4a(y_1 - y_2)$$

$$\text{and} \quad l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\therefore \quad (x_1^2 - x_2^2)^2 = 16a^2[l^2 - (x_1 - x_2)^2]$$

$$\therefore \quad 2(x_1^2 - x_2^2) \cdot 2\left(x_1 - x_2 \frac{dx_2}{dx_1}\right) = -32a^2(x_1 - x_2)\left(1 - \frac{dx_2}{dx_1}\right)$$

$$\therefore \quad \frac{dx_2}{dx_1} = \frac{x_1(x_1 + x_2) + 8a^2}{x_2(x_1 + x_2) + 8a^2}. \quad \dots \quad \dots \quad (2)$$

From equations (1) and (2), we have

$$\begin{aligned} \frac{d\bar{y}}{dx_1} &= \frac{1}{4a} \left[ \frac{2x_1x_2(x_1 + x_2) + 8a^2(x_1 + x_2)}{x_2(x_1 + x_2) + 8a^2} \right] \\ &= \frac{(x_1 + x_2)(x_1x_2 + 4a^2)}{2a[x_2(x_1 + x_2) + 8a^2]}. \quad \dots \quad \dots \quad (3) \end{aligned}$$

The rod is therefore in equilibrium when  $x_1 + x_2 = 0$  and when  $x_1x_2 + 4a^2 = 0$ , i.e. when the rod is horizontal and when it passes through the focus.

$$\text{From (3),} \quad \frac{d^2\bar{y}}{dx_1^2} =$$

$$\frac{x_1 + x_2}{2a} \cdot \frac{d}{dx_1} \left[ \frac{x_1x_2 + 4a^2}{x_2(x_1 + x_2) + 8a^2} \right] + \left[ \frac{x_1x_2 + 4a^2}{x_2(x_1 + x_2) + 8a^2} \right] \left( \frac{1}{2a} + \frac{dx_2}{dx_1} \right).$$

$$\text{When} \quad x_1 + x_2 = 0, \quad x_1 = \frac{l}{2}, \quad x_2 = -\frac{l}{2}, \quad \frac{dx_2}{dx_1} = 1,$$

so that  $\frac{d^2\bar{y}}{dx^2} = \frac{-\frac{l^2}{4} + 4a^2}{8a^2}$  which is  $+ve$  or  $-ve$  according as  $l \lesseqgtr 4a$ .

Thus in the horizontal position, the equilibrium is stable or unstable according as the length of the rod is less or greater than the latus rectum of the parabola.

NOTE 5. The equation of any line through the focus is  $y - a = mx$ .

This line cuts the parabola at points whose abscissae are given by  $\frac{x^2}{4a} - a = mx$ ; i.e.  $x^2 - 4amx - 4a^2 = 0$ . The product of the roots,  $x_1x_2$  is  $-4a^2$  so that if  $x_1, x_2 + 4a^2 = 0$ , the line passes through the focus.

*Alternative Method.*—Suppose the line of the rod makes an angle  $\theta$  with  $Ox$  and cuts it at  $C$  where  $OC = c$ . If  $r$  represents the distance of any point on the line from  $C$ , the co-ordinates of this point are given by  $x = r \cos \theta - c$ ,  $y = r \sin \theta$ .

The values of  $r$  ( $r_1$  and  $r_2$ ) of the points  $A$  and  $B$ , are the roots of the equation

$$\begin{aligned} \text{i.e. of } & (r \cos \theta - c)^2 = 4ar \sin \theta \\ & r^2 \cos^2 \theta - 2r(c \cos \theta + 2a \sin \theta) + c^2 = 0 \\ \therefore & \bar{y} = \frac{1}{2}(r_1 + r_2) \sin \theta = \frac{\sin \theta (c \cos \theta + 2a \sin \theta)}{\cos^2 \theta} \end{aligned}$$

$$\text{i.e. } \bar{y} = c \tan \theta + 2a \tan^2 \theta. \quad \dots \dots \dots (1)$$

$$\text{Also } l^2 = (r_1 - r_2)^2 = \frac{4(c^2 \cos^2 \theta + 4ac \sin \theta \cos \theta + 4a^2 \sin^2 \theta) - 4c^2 \cos^2 \theta}{\cos^4 \theta}$$

$$\therefore \frac{l^2}{4} = \frac{4a \tan \theta}{\cos^2 \theta} (c + a \tan \theta). \quad \dots \dots \dots (2)$$

Eliminating  $c$  from equations (1) and (2) we have

$$\bar{y} = a \tan^2 \theta + \frac{l^2}{16a} \cos^2 \theta \quad \dots \dots \dots (3)$$

$$\begin{aligned} \therefore \frac{d\bar{y}}{d\theta} &= 2a \tan \theta \cdot \sec^2 \theta - \frac{l^2}{8a} \cos \theta \sin \theta \\ &= \frac{\sin \theta}{\cos^3 \theta} \left[ 2a - \frac{l^2}{8a} \cos^4 \theta \right]. \quad \dots \dots \dots (4) \end{aligned}$$

$\therefore$  there is equilibrium when  $\theta = 0$  and when  $\cos^2 \theta = \frac{4a}{l^2}$ .

$$\text{From (4), } \frac{d^2\bar{y}}{d\theta^2} = \sin \theta \cdot \frac{d}{d\theta} \left[ \frac{2a}{\cos^3 \theta} - \frac{l^2}{8a} \cos \theta \right] + \cos \theta \left[ \frac{2a}{\cos^3 \theta} - \frac{l^2}{8a} \cos \theta \right].$$

When  $\theta = 0$ ,  $\frac{d^2\bar{y}}{d\theta^2} = 2a - \frac{l^2}{8a}$  which is  $+ve$  or  $-ve$  according as  $l \lesseqgtr 4a$ .

**46. Stability of a Body Resting on a Perfectly Rough Surface.** A body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvature  $\rho_1$  and  $\rho_2$  respectively. The centre of gravity of the first body is at a height  $h$  above the point of contact and the common normal makes angle  $\alpha$  with the vertical. It is required to prove that the equilibrium is stable if  $h < \frac{\rho_1 \rho_2 \cos \alpha}{\rho_1 + \rho_2}$ .

$O_1$  and  $O_2$  are the centres of curvature,  $G$  is the centre of gravity of the upper body, and  $A$  is the point of contact when there is equilibrium. Let  $O_1G = h$  and  $\widehat{GO_1A} = \beta$

Suppose the upper body to be rotated through a small angle  $\phi$  thus causing the line  $O_2O_1$  to rotate through a small angle  $\theta$ . B is the new point of contact and  $A_1$ ,  $A_2$  the points

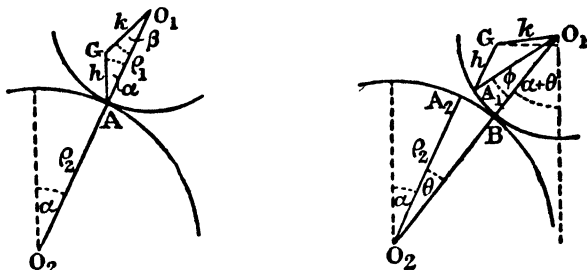


Fig. 83.

formerly in contact. Then  $O_2O_1$ ,  $A_1O_1$ ,  $A_1G$  and  $GO_1$  make with the vertical, angles of  $(\alpha + \theta)$ ,  $(\alpha + \theta + \phi)$ ,  $\phi$  and  $(\alpha + \theta + \phi + \beta)$  respectively.

If  $\bar{y}$  be the height of G above  $O_2$ , then  
 $\bar{y} = (\rho_1 + \rho_2) \cos(\alpha + \theta) - k \cos(\alpha + \theta + \phi + \beta) \quad \dots (1)$

As  $\text{arc } A_2B = \text{arc } A_1B$ ,  $\rho_1\phi = \rho_2\theta \quad \therefore \quad \frac{d\phi}{d\theta} = \frac{\rho_2}{\rho_1}$

$$\therefore \quad \frac{d\bar{y}}{d\theta} = -(\rho_1 + \rho_2) \sin(\alpha + \theta) + k \sin(\alpha + \theta + \phi + \beta) \cdot \left(1 + \frac{\rho_2}{\rho_1}\right)$$

$$= \frac{(\rho_1 + \rho_2)}{\rho_1} \left[ k \sin(\alpha + \theta + \phi + \beta) - \rho_1 \sin(\alpha + \theta) \right] \dots (2)$$

When  $\theta = 0$  and  $\phi = 0$ , there is equilibrium and consequently  
 $k \sin(\alpha + \beta) - \rho_1 \sin \alpha = 0 \quad \dots \dots (3)$

$$\text{Now } \frac{d^2\bar{y}}{d\theta^2} = \frac{\rho_1 + \rho_2}{\rho_1} \left[ k \cos(\alpha + \theta + \phi + \beta) \cdot \left(1 + \frac{\rho_2}{\rho_1}\right) - \rho_1 \cos(\alpha + \theta) \right] \dots \dots (4)$$

which, when  $\theta = \phi = 0$ , becomes

$$\frac{\rho_1 + \rho_2}{\rho_1^2} \left[ k \cos(\alpha + \beta) \cdot (\rho_1 + \rho_2) - \rho_1^2 \cos \alpha \right]$$

Thus, in order that equilibrium should be stable,

$$k(\rho_1 + \rho_2) \cos(\alpha + \beta) > \rho_1^2 \cos \alpha.$$

Now from the  $\Delta AGO_1$ ,  $h = \rho_1 \cos \alpha - k \cos(\alpha + \beta)$ .

$\therefore$  the above condition becomes

$$(\rho_1 + \rho_2)(\rho_1 \cos \alpha - h) > \rho_1^2 \cos \alpha.$$

i.e.

$$h < \frac{\rho_1 \rho_2 \cos \alpha}{\rho_1 + \rho_2}.$$

**Ex. XIII.** A solid of uniform density in the form of a portion of a paraboloid of revolution bounded by a plane perpendicular to the axis is placed with its vertex resting upon a horizontal plane, and with the axis vertical. Determine the height of the solid in order that the equilibrium should be stable.

Let the equation of a section through the axis of the paraboloid of revolution be  $x^2 = 4ay$ , and let  $H$  be the height.

Then the height of the C.G. is  $\frac{3}{8}H$ .

The radius of curvature at the vertex is  $2a$ .

Comparing with the notation of the previous problem we have  $\rho_1 = 2a$ ,  $\rho_2 = \infty$ ,  $\alpha = 0$  and  $h = \frac{3}{8}H$ , so that

$$\frac{\rho_1 \rho_2 \cos \alpha}{\rho_1 + \rho_2} = \frac{2a}{\frac{2a}{\rho_2} + 1} = 2a.$$

Thus the equilibrium is stable provided that

$$\frac{3}{8}H < 2a$$

i.e.

$$H < 3a.$$

### Problems VI.

1. Find the least force required to move a body of mass  $M$  up an incline  $\alpha$ , the co-efficient of friction being  $\mu$ .

2. A disc, radius  $r$  and weight  $W$ , rests on a rough horizontal table whose co-efficient of friction is  $\mu$ . It is free to rotate about a smooth peg at its centre. What is the least force applied to a point on the circumference that is required to rotate the disc?

3. A uniform rod lying on an inclined plane of slope  $\alpha$ , can rotate about a point in it at distances  $a$  and  $b$  from its ends. Show that, if  $\mu$  be the co-efficient of friction, the inclination of the rod to the line of greatest slope cannot be greater than

$$\sin^{-1} \left\{ \mu \cot \alpha \left( \frac{a^2 + b^2}{a^2 - b^2} \right) \right\}.$$

4. A heavy plate in the form of a square lies on a rough plane which is inclined to the horizon at an angle  $\alpha$ , and the plate can turn about a pin at one of its corners. Assuming that the weight of the plate is equally distributed over its area, prove that it will rest in any position if

$$\frac{3}{\mu} \tan \alpha < 2 + \sqrt{2} \log (1 + \sqrt{2})$$

where  $\mu$  is the co-efficient of limiting friction.

5. A particle rests on a smooth ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  acted on by forces  $\lambda x^n$ ,  $\mu y^n$  parallel to the axes of  $x$  and  $y$  respectively; find its position of equilibrium. Explain the case in which  $n = 1$ .

6. A heavy particle rests on a rough parabola, axis vertical, vertex downwards, latus rectum  $4a$ . Find the greatest altitude (above the vertex) at which the particle could remain at rest. If the greatest altitude be  $h$ , what is the co-efficient of friction?

7. A rough wire (co-efficient of friction,  $\mu$ ) is in the shape of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , the axis of  $x$  being vertical and the axis of  $y$  horizontal. A bead rests on the wire : find the distance below the highest point, of the position of limiting equilibrium.

8. A rough wire is in the form of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . A small bead on the wire is in limiting equilibrium when at a distance  $\frac{b}{2}$  from the major axis under the action of a force whose line of action passes through the centre of the ellipse. Prove that  $\sqrt{3}(a^2 - b^2) < 4ab$  and show that when  $a = 3$  and  $b = 2$ , the co-efficient of friction is  $\frac{5\sqrt{3}}{24}$ .

9. Two equal beams AB, AC, each of weight W, connected by a hinge at A, are placed in a vertical plane with their extremities B, C resting on a horizontal plane ; they are kept from falling by a stay connecting B with the middle point of AC.

Show by the principle of virtual work that the tension in the stay is  $\frac{W}{4} \sqrt{1 + 9 \cot^2 \theta}$ , where  $\theta$  is the inclination of either beam to the horizontal.

10. Four equal jointed rods, each of length 2 feet, are hung from an angular point which is connected by an elastic string with the opposite point. The rods hang in the form of a square. Find the tension of the string and prove that, if the modulus of elasticity be equal to the weight of a rod, the unstretched length of the string is  $\frac{2\sqrt{2}}{3}$  feet.

11. A triangle ABC of any shape is formed of light rods smoothly jointed to each other at their ends. It is placed in a vertical plane with A downwards and the rods AB, AC resting on two smooth pegs in a horizontal line. A weight W is suspended from A ; prove that the stress in the rod BC is

$$\frac{1}{2} \frac{Wl}{p} \cos^2 \frac{1}{2} A,$$

where  $2l$  is the distance between the pegs and  $p$  is the perpendicular from A on BC.

12. Six uniform bars, jointed together, hang from a fixed point A, and form a regular hexagon ABCDEF, which is kept in shape by light horizontal struts BF, CE. Prove that the thrusts in these are in the ratio 5 : 1.

13. A quadrilateral ABCD, in which  $AB = BC$  and  $CD = DA$ , is formed of four uniform rods, each of weight W, jointed together : the quadrilateral is hung up from the point B, and the joints A, C are kept apart by a weightless rod. Show that the thrust in this rod is  $\frac{W}{2}(3 \tan \alpha + \tan \beta)$  where  $\alpha, \beta$  are the angles which the upper and lower rods respectively make with the vertical.

14. Five equal uniform heavy rods freely jointed at their ends form a regular pentagon  $ABCDE$  and  $BE$  is joined by a weightless bar. The system is suspended from  $A$  in a vertical plane; prove that the thrust in  $BE$  is  $W \cot \frac{\pi}{10}$ , where  $W$  is the weight of a rod.

15. A hexagon of smoothly jointed rods  $ABCDEF$  is suspended from the point  $A$  and is kept in the form of a regular hexagon by two struts  $BF$  and  $CE$ . If the weight of each rod is  $W$  and that of each strut  $w$ , prove that the thrust in the strut  $BF$  is  $\frac{\sqrt{3}}{2}(5W + 2w)$ .

16. Three similar uniform rods  $AB$ ,  $BC$ ,  $CD$  each of length  $a$  are smoothly jointed together at  $B$  and  $C$  and the ends  $A$  and  $D$  are smoothly jointed at points on the same level at a distance  $2a$  apart. A weight  $W$  hangs from a point of trisection of  $BC$  and the system is kept in equilibrium with  $BC$  horizontal by a force  $P$  acting along  $BC$ . Prove that  $3\sqrt{3}P = W$ , independently of the weight of the rods.

17. A rhombus  $ABCD$  is formed of four light rods smoothly jointed at their ends.  $A$  is connected with  $C$ , and  $B$  with the middle point of  $CD$ , by tight strings. Show that if these strings be of equal length the tension of one of them is double that of the other.

18. Twelve equal uniform rods form a cube having universal joints at each of its angles. Show that, if it be suspended by one of its angles and be prevented from collapsing by three rods without weight forming the diagonals not passing through the point of suspension, the thrust along the three rods will be half the weight of the framework.

19. An isosceles wedge, whose vertical angle is  $\alpha$ , rests with its vertex downwards between two smooth parallel horizontal rails, on the same level at a distance  $b$  apart; show that if the distance  $c$  from the centre of gravity to the vertex is less than  $2b \operatorname{cosec} \alpha$ , the vertical position of equilibrium is stable, and that unstable oblique positions of equilibrium are possible.

20.  $ABCDE$  is a plane lamina in the form of a rectangle with a semicircle  $AED$  attached to the side  $AD$ ,  $E$  being the middle point of the arc  $AED$ . The lamina is balanced vertically with the vertex  $E$  in contact with a horizontal plane. Show that the equilibrium is unstable if the ratio  $AB : AD$  is greater than  $\frac{1}{\sqrt{6}}$ .

21. A rough plank of thickness  $2b$  is laid across a fixed cylinder of radius  $a$  and rests in equilibrium at an angle  $\alpha$  with the horizontal. By finding the increase of potential energy when the plank is rolled through an angle  $\theta$ , show that the equilibrium is stable if  $b < a \cos^2 \alpha$ .

22. A uniform rod rests with its ends on a smooth parabolic wire, whose axis is vertical and vertex downwards. Show that, if the length of the rod exceeds the latus rectum, a position of equilibrium is possible in which the rod passes through the focus, and prove that it is stable.

23. A uniform solid hemisphere is placed with its curved surface in contact with a rough inclined plane. Show that for equilibrium to be possible, the inclination of the plane to the horizontal must be less than  $\sin^{-1}(3/8)$ . Show also, that there are two positions of equilibrium if the inclination of the plane lies between  $\tan^{-1}(3/8)$  and  $\sin^{-1}(3/8)$ . Show that when one position only exists it is stable.

24. A uniform hemispherical shell rests upon an inclined plane. Show that for equilibrium the inclination of the plane to the horizon must not be more than  $30^\circ$  and that the equilibrium is stable.

25. Two heavy rings slide on a fixed smooth parabolic wire whose axis is horizontal, and the rings are connected by a string which passes over a smooth peg at the focus. Prove that in the position of equilibrium the depths of the rings below the axis of the parabola are proportional to their weights.

26. Two particles are connected by a fine string and can move freely in a smooth cycloidal tube whose vertex is upwards, the string passing over the vertex. Prove that in equilibrium the arcual distances of the particles from the vertex must be inversely as their masses.

27. Three equal uniform rods AB, BC, CD, freely jointed at B and C have small smooth weightless rings attached to them at A and D. The rings slide on a smooth parabolic wire whose axis is vertical and vertex upwards, and whose latus rectum is half the length of the three rods. Prove that in the position of equilibrium the inclination  $\theta$  of AB or CD to the vertical is given by the equation  $\cos \theta - \sin \theta + \sin 2\theta = 0$ .

28. A smooth wire is in the form of the cardioid  $r = a(1 - \cos \theta)$  with the axis vertical. A string of length  $l$  has masses  $W$  and  $2W$  attached to the ends such that the latter rests on the wire and the former hangs vertically, the string passing over a smooth peg at the cusp. Show that there is stable equilibrium when the portion of the string from the cusp to the mass  $2W$  is inclined at  $\cos^{-1} \frac{1}{2}$  to the vertical.

29. A uniform solid consists of a cone and a hemisphere of equal bases fastened together so that the bases coincide. Show that if the vertical angle of the cone is greater than  $60^\circ$ , the solid will always move to an upright position if placed with the surface of the hemisphere on a horizontal plane.

30. Two equal uniform rods AC, BC are smoothly hinged to each other at C and carry light rings at the ends A and B. The rings slide on a smooth parabolic wire which is fixed in a vertical plane with the vertex uppermost. Prove that there is a position of equilibrium in which each rod is inclined to the vertical at an angle  $\cos^{-1}(l/2a)$ , where  $l$  is the semi-latus rectum of the parabola and  $a$  the length of the rod.

31. An elliptic cylinder rests in stable equilibrium on a horizontal plane. Prove that the equilibrium cannot be made unstable by placing a particle on the highest generator if the eccentricity of the

cross section is greater than  $\frac{1}{\sqrt{2}}$ .



32. A uniform hemisphere rests in equilibrium with its base upwards on top of a sphere of double its radius. Show that the greatest weight which can be placed at the centre of the plane face without rendering the equilibrium unstable is one-eighth of the weight of the hemisphere.

33. An open cylindrical can whose height is 1 ft. and diameter 1 ft. is poised on the top of a sphere. Show that the least diameter of the sphere consistent with stability is 9.6 in.

34. An elliptic cylinder rests with its curved surface in contact with two smooth planes each inclined at an angle of  $45^\circ$  to the horizontal. Show that the position of equilibrium where the axes of the cross section are horizontal and vertical is unstable whereas the position where the axes are parallel to the planes is one of stable equilibrium.

35. A fixed spherical shell has a small hole in it at an angular distance  $\alpha$  from the highest point. A uniform rod whose length is four times the radius of the shell passes through the hole and rests with one end in contact with the inner surface of the shell, all the surfaces being smooth. Write down the potential energy of the rod when it is in the same vertical plane as the highest point of the shell, and makes an angle  $\theta$  with the horizontal.

If  $\alpha = 30^\circ$ , show that the acute angle which the rod makes with the horizontal in equilibrium may be  $20^\circ$ ,  $40^\circ$ , or  $60^\circ$ .

36. The diagram shows a simple engine mechanism. The point A is pushed downwards by a constant vertical force  $W$ , and is free to move smoothly in the vertical line CD which passes through O. The crank OB which rotates about O is prolonged to E, where a weight  $w$  is attached. Neglecting the weights of the links AB, EB, show that the positions of equilibrium are given by

$$\theta = 0, \theta = \pi, \theta = \pm \theta_1, \theta = \pi \pm \theta_1,$$

$$\text{where } \sin \theta_1 = \sqrt{\frac{l^2(aw - Wr)^2 - W^2r^4}{r^2aw(aw - 2Wr)}}.$$

Here  $AB = l$ ,  $OB = r$ ,  $OE = a$ , and  $\theta$  is the angle AOB.

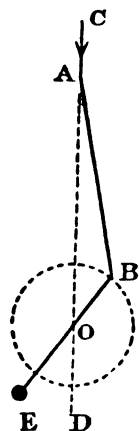


Fig. 84.

37. A fixed elliptic wire has its major axis vertical and a uniform rod of length  $2l$  moves on it, being attached by smooth rings at its ends. Show that if  $2\alpha$  be the sum of the eccentric angles of its ends, the height of its centre of gravity above the centre of the ellipse is

$$a \cos \alpha \left[ 1 - \frac{l^2}{a^2(1 - e^2 \cos^2 \alpha)} \right]^{\frac{1}{2}}.$$

Hence show that, if the rod lie between the latus rectum and the minor axis of the ellipse in length, the horizontal position of equilibrium in which the rod is above the centre is stable, and that in which it is below is unstable.

## CHAPTER VII

### MOMENTS OF INERTIA : KINETIC ENERGY OF ROTATING BODIES

**47. Definition of Moment of Inertia.** If the mass of every particle of a material system be multiplied by the square of its distance from a straight line, the sum of the products so formed is called the **moment of inertia** of the system about that line.

If the moment of inertia of a system about a given straight line be  $Mk^2$  where  $M$  is the whole mass, the length  $k$  is called the **radius of gyration** of the system about that line.

#### 48. Moment of Inertia about Perpendicular Axes.

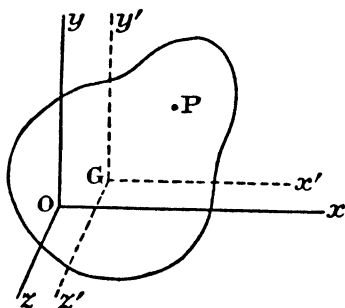


Fig. 85.

Let  $Ox$  and  $Oy$  be the axes and consider a **lamina** in the plane  $xOy$ . Let  $x, y$  be the co-ordinates of any particle  $P$  whose mass is  $m$ . Then if  $I_x, I_y$  be the moments of inertia of the lamina about  $Ox$  and  $Oy$ ,  $I_x = \sum my^2$  and  $I_y = \sum mx^2$ . Now  $I_x + I_y = \sum my^2 + \sum mx^2 = \sum m(x^2 + y^2)$  by combining the terms of the summation where the masses are the same.

If  $r$  be the distance of  $P$  from  $O$  and  $I_z$  be the moment of inertia of the lamina about a line ( $Oz$ ) perpendicular to the plane  $xOy$ ,  $I_z = \sum mr^2 = \sum m(x^2 + y^2)$   
 $\therefore I_z = I_x + I_y$ .

Consider a right prism whose right section is as shown, and suppose  $I_z = m_1 k^2$  where  $m_1$  is the mass of the lamina and consequently,  $k$  the radius of gyration about  $Oz$ . The prism is formed by a number of laminae similarly situated as regards the line  $Oz$ . The radius of gyration of every lamina about  $Oz$  is  $k$  and thus the moment of inertia of the prism about  $Oz$  is  $Mk^2$  where  $M$  is the mass of the prism.

In general, for a solid body,

$$I_x = \sum m(y^2 + z^2), I_y = \sum m(z^2 + x^2) \text{ and } I_z = \sum m(x^2 + y^2).$$

$$\text{Consequently, } I_x + I_y + I_z = 2 \sum mr^2.$$

As an example on the last result, consider a thin spherical shell of mass  $M$  and radius  $a$ . If  $O$  be taken as the centre of the sphere, by symmetry  $I_x = I_y = I_z$  and  $\sum mr^2 = Ma^2$ . Thus  $I_x = \frac{2}{3}Ma^2$ , which is the moment of inertia about a diameter.

**49. Theorem of Parallel Axes.**

Given the moment of inertia of a body about an axis through the centre of gravity, find the moment of inertia about a parallel axis.

Referring to Fig. 85 let  $G$  be the centre of gravity of the body and  $Gz'$  and  $Oz$  the two parallel axes. Let  $P$  be a particle whose co-ordinates referred to rectangular axes through  $O$  and  $G$  are respectively  $x, y, z$ , and  $x', y', z'$ . Also let the co-ordinates of  $G$  referred to the axes  $Ox, Oy, Oz$  be  $\bar{x}, \bar{y}, \bar{z}$ .

If  $I_1$  be the moment of inertia of the body about  $Oz$ , then

$$\begin{aligned} I_1 &= \sum m(x^2 + y^2) \\ &= \sum m[(\bar{x} + x')^2 + (\bar{y} + y')^2] \\ &= (\bar{x}^2 + \bar{y}^2) \cdot \sum m + \sum m(x'^2 + y'^2) + 2\bar{x} \sum mx' + 2\bar{y} \sum my'. \end{aligned}$$

Now  $\sqrt{\bar{x}^2 + \bar{y}^2}$  is the distance,  $d_1$ , between the axes  $Gz'$  and  $Oz$ , and  $\sum m(x'^2 + y'^2)$  is the moment of inertia of the body about  $Gz'$ . Also, the distances of the centre of gravity of the body from the planes  $z'Gx'$  and  $y'Gz'$  are  $\frac{\sum my'}{\sum m}$  and  $\frac{\sum mx'}{\sum m}$  respectively, but as  $G$  is the centre of gravity,  $\sum mx'$  and  $\sum my'$  are zero.

If  $\sum m(x'^2 + y'^2) = Mk^2$  where  $M$  is the mass of the body, the above result becomes

$$I_1 = Mk^2 + Md_1^2.$$

Thus the moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis through the centre of gravity together with the product of the mass and the square of the distance between the axes.

If  $I_2$  be the moment of inertia about another parallel axis at distance  $d_2$  from  $Gz'$

$$I_2 = Mk^2 + Md_2^2$$

so that

$$I_1 - I_2 = M(d_1^2 - d_2^2).$$

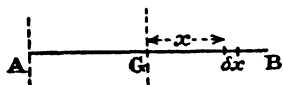
**50. Moments of Inertia about Various Axes of Bodies Having Certain Simple Geometrical Shapes.****I. UNIFORM THIN ROD.**

Fig. 86.

to the rod.

Let  $l \equiv$  length,  $m \equiv$  mass per unit length,  $M \equiv$  whole mass and  $Mk^2 \equiv$  moment of inertia about an axis through the centre  $G$  at right angles

$$\text{Then} \quad Mk^2 = m \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = m \frac{l^3}{12} = M \frac{l^2}{12}.$$

By Article 49, the moment of inertia about a parallel axis through one end

$$= M \left[ \frac{l^2}{12} + \frac{l^2}{4} \right] = M \frac{l^2}{3}.$$

## II. RECTANGULAR LAMINA.

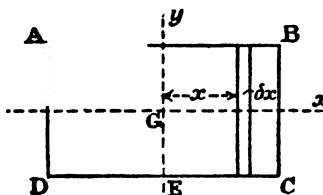


Fig. 87.

$AB = l$ ,  $BC = b$ ,  $m \equiv$  mass per unit area,  $M \equiv$  whole mass. Let the moments of inertia about axes through the C.G. parallel to the length and breadth be  $Mk_1^2$  and  $Mk_2^2$  respectively.

Then

$$Mk_2^2 = mb \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = \frac{mb l^3}{12} = \frac{M l^3}{12}.$$

Similarly

$$Mk_1^2 = M \frac{b^3}{12}.$$

By Article 48, the moment of inertia about an axis through G perpendicular to the lamina is  $M \frac{l^2 + b^2}{12}$ .

By Article 49, the moment of inertia about AB is

$$M \left[ \frac{b^2}{12} + \frac{b^2}{4} \right] = M \frac{b^2}{3}.$$

Also, the moments of inertia about axes through E the middle point of CD and through B perpendicular to the lamina are respectively

$$M \left[ \frac{l^2 + b^2}{12} + \frac{b^2}{4} \right] \text{ and } M \left[ \frac{l^2 + b^2}{12} + \frac{l^2 + b^2}{4} \right]$$

i.e.  $M \left[ \frac{l^2}{12} + \frac{b^2}{3} \right] \text{ and } M \frac{l^2 + b^2}{3}.$

If a right rectangular prism have dimensions  $l \times b \times h$ , the moments of inertia about axes through the centre parallel to the edges are  $M \frac{b^2 + h^2}{12}$ ,  $M \frac{l^2 + h^2}{12}$  and  $M \frac{l^2 + b^2}{12}$ .

Also the moments of inertia about the edges are

$$M \frac{b^2 + h^2}{3}, M \frac{l^2 + h^2}{3} \text{ and } M \frac{l^2 + b^2}{3}.$$

## III. TRIANGULAR LAMINA.

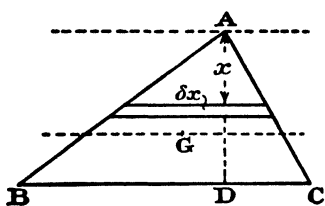


Fig. 88.

AD is perp. to BC. BC =  $b$  and AD =  $h$ .  $m \equiv$  mass per unit area,  $M \equiv$  whole mass.

Let the moments of inertia about axes through the centroid G and through A parallel to BC be  $Mk^2$  and  $Mk_1^2$  respectively.

$$\text{Then } Mk_1^2 = m \cdot \int_0^h x^2 dx = m \cdot \frac{bh^3}{4} = M \frac{h^2}{2}$$

$$\therefore Mk^2 = M \left[ \frac{h^2}{2} - \frac{4h^2}{9} \right] = M \cdot \frac{h^2}{18}.$$

$$\text{The moment of inertia about BC} = M \left[ \frac{h^2}{18} + \frac{h^2}{9} \right] = M \frac{h^2}{6}.$$

## IV. CIRCULAR DISC.

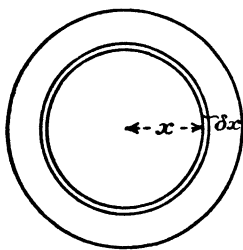


Fig. 89.

Let  $a \equiv$  radius,  $m \equiv$  mass per unit area,  $M \equiv$  total mass.

Let the moment of inertia of the disc about an axis through its centre be  $Mk^2$ .

$$\text{Then } Mk^2 = 2\pi m \int_0^a x^3 dx = \frac{\pi m a^4}{4} = M \cdot \frac{a^2}{2}.$$

$\therefore$  the sum of the moments of inertia about two diameters at right angles is

$$M \frac{a^2}{2}; \therefore \text{ the moment of inertia about}$$

$$\text{a diameter is } M \frac{a^2}{4}.$$

The moment of inertia of a circular cylinder about its axis is  $M \frac{a^2}{2}$ .

V. SPHERE. The moment of inertia of a spherical shell, radii  $x$ ,  $x + \delta x$  and mass  $m$  per unit volume, about a diameter is  $4\pi x^2 \cdot \delta x \cdot m \cdot \frac{3}{2}x^2 = \frac{3}{2}\pi m x^4 \delta x$ . Therefore the moment of inertia of a solid sphere, radius  $a$ , about a diameter is

$$\frac{3}{2}\pi m \int_0^a x^4 dx = \frac{3}{10}\pi m a^5 = M \cdot \frac{2}{5}a^2, \text{ where } M \text{ is the mass of the sphere.}$$

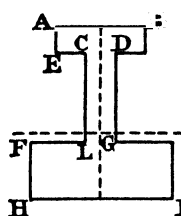
**Ex. I.**


Fig. 90.

The figure represents the cross section of a girder:  $AB = CL = 6''$ ,  $AE = CD = 2''$ ,  $FH = 4''$ ,  $HK = 10''$ . Find the numerical value of  $Ak^2$  where  $A$  is the whole area and  $k$  is the radius of gyration about an axis through the centroid parallel to  $HK$ .

Let  $\bar{x}$  in. be the distance of the centroid from  $HK$ , then  $Ak^2 = 12 \times (\frac{1}{3} + 12\bar{x}) + 12 \times (3 + 4\bar{x}) + 40 \times (\frac{1}{3} + 4) - 64\bar{x}^2$   
 $= 2293\frac{1}{3} - 64\bar{x}^2$  in.<sup>(4)</sup>

Also  $64\bar{x} = 12 \times 11 + 12 \times 7 + 40 \times 2 = 296$

$\therefore Ak^2 = 2293\frac{1}{3} - \frac{296 \times 296}{64} = 924\frac{1}{3}$  in.<sup>(4)</sup>.

**Ex. II.** Find the moment of inertia of a right circular cylinder, radius  $a$  and length  $l$  about a line at right angles to its axis and passing through the middle point.

The moment of inertia about the line of a disc whose centre is at distance  $x$  from the middle point of the axis is  $\pi a^2 \cdot \delta x \cdot m \cdot (\frac{a^2}{4} + x^2)$  where  $m$  is the density of the cylinder.

$\therefore$  moment of inertia of the cylinder about the line

$$= \pi a^2 m \int_{-\frac{l}{2}}^{\frac{l}{2}} \left( \frac{a^2}{4} + x^2 \right) dx.$$

$$= \pi a^2 l m \cdot \left( \frac{a^2}{4} + \frac{l^2}{12} \right)$$

$$= M \cdot \frac{3a^2 + l^2}{12}.$$

**Ex. III.** Find the moment of inertia of a thin wire in the form of an arc of a circle whose radius is  $a$  and which subtends an angle  $2a$  at the centre  $O$  about an axis (i.) through its middle point perpendicular to its plane, (ii.) coinciding with the diameter

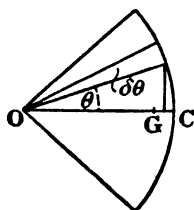


Fig. 91.

(i.) If  $G$  is the centroid,  $OG = \frac{a \sin a}{a}$  and

$$GC = a \left( 1 - \frac{\sin a}{a} \right).$$

Moment of inertia about an axis through  $O$  perp. to the plane  $= Ma^2$

$\therefore$  moment of inertia about an axis through  $C$  perp. to the plane

$$\begin{aligned}
 &= Ma^2 - M \frac{a^3 \sin^2 \alpha}{a^2} + Ma^2 \left(1 - \frac{\sin \alpha}{a}\right)^2 \\
 &= 2Ma^2 \left(1 - \frac{\sin \alpha}{a}\right).
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii.) Moment of inertia about OC} &= ma^3 \int_{-\alpha}^{\alpha} \sin^2 \theta d\theta \\
 &= \frac{1}{2} ma^3 \int_{-\alpha}^{\alpha} (1 - \cos 2\theta) d\theta \\
 &= ma^3 \left(\alpha - \frac{1}{2} \sin 2\alpha\right)
 \end{aligned}$$

Now  $2a\alpha \cdot m = M$  so that  $m = \frac{M}{2a\alpha}$ .

$\therefore$  moment of inertia about OC  $= M \cdot \frac{a^2}{2} \left(1 - \frac{\sin 2\alpha}{2\alpha}\right)$ .

**Ex. IV.** Find the moment of inertia of a triangle about a line through the centroid at right angles to the plane.

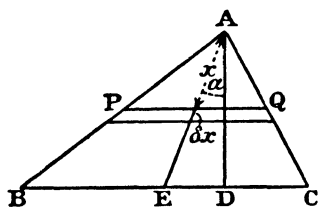


Fig. 92.

ABC is the triangle which has surface density  $m$ . AE is a median and AD is perp. to BC. Let  $\widehat{EAD} = \alpha$  and  $AE = d$  so that  $AG = \frac{2}{3}d$ .

Consider an element PQ, parallel to BC, having its middle point at distance  $x$  from A.

Its mass is  $\frac{ax}{d} \cdot \delta x \cos \alpha \cdot m$ , and its moment of inertia about a line through A perp. to the plane is

$$\frac{max \cos \alpha \cdot \delta x}{d} \cdot \left[ \frac{a^2 x^2}{12d^2} + x^2 \right].$$

$\therefore$  moment of inertia of the triangle about this line

$$= \frac{ma \cos \alpha}{d} \int_0^d \left( \frac{a^2 x^3}{12d^2} + x^3 \right) dx.$$

$$= mad \cos \alpha \cdot \left( \frac{a^2}{48} + \frac{d^2}{4} \right).$$

Now if  $M$  be the whole mass,  $M = \frac{1}{3}mad \cos \alpha$ .

$\therefore$  moment of inertia of the triangle about the line through G perpendicular to the plane is

$$\begin{aligned}
 M \left[ \left( \frac{a^2}{24} + \frac{d^2}{2} \right) - \frac{4d^2}{9} \right] &= M \left[ \frac{3a^2 + 4d^2}{72} \right] \\
 &= M \cdot \left[ \frac{3a^2 + (2b^2 + 2c^2 - a^2)}{72} \right] = M \cdot \left[ \frac{a^2 + b^2 + c^2}{36} \right].
 \end{aligned}$$

**Ex. V.** Find the moment of inertia about its axis of an anchor ring, mean radius  $R$ , radius of section  $r$ , and mass  $M$ .

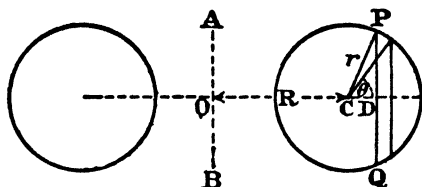


Fig. 93.

The mass of the element determined by  $PQ$  ( $CD = x$ ) is  $2\pi(R+x) \cdot 2\sqrt{r^2-x^2} \cdot \delta x \cdot m$ , where  $m$  is the density.

$\therefore$  moment of inertia of anchor ring about  $AOB$ , is

$$\begin{aligned}
 & 4\pi m \int_{-r}^r (R+x)^2 \cdot \sqrt{r^2-x^2} \cdot dx \\
 &= -4\pi m r^2 \int_{\pi}^0 (R+r \cos \theta)^2 \cdot \sin^3 \theta \cdot d\theta. \\
 &= 4\pi m r^2 \left[ \int_0^{\pi} \left\{ \frac{R^3}{2} (1 - \cos 2\theta) + 3R^2 r \cos \theta \sin^3 \theta + \frac{3Rr^3}{8} (1 - \cos 4\theta) \right. \right. \\
 & \quad \left. \left. + r^3 \cos^3 \theta \sin^3 \theta \right\} d\theta \right] \\
 &= 4\pi m r^2 \left[ \frac{R^3 \theta}{2} - \frac{R^3 \sin 2\theta}{4} + R^2 r \sin^3 \theta + \frac{3Rr^2 \theta}{8} - \frac{3Rr^2 \sin 4\theta}{32} \right. \\
 & \quad \left. + \frac{r^3 \sin^3 \theta}{3} - \frac{r^3 \sin^5 \theta}{5} \right]_{\pi}^0 \\
 &= \frac{\pi^2 m R r^2}{2} (4R^3 + 3r^3) \\
 &= M \cdot \left[ \frac{4R^3 + 3r^3}{4} \right].
 \end{aligned}$$

**51. Equipomental Systems of Particles.** Two bodies or two systems of particles are said to be equipomental when their moments of inertia about all straight lines are equal each to each. Given any direction, the line about which the moment of inertia of a system of particles is least is that passing through the centre of gravity, and as this is of course least for an equipomental system, the line also passes through the centre of gravity of the latter. The same applies for any other direction, from which it follows that the centres of gravity of two equipomental systems of bodies must coincide.

Consider the case of a triangle  $ABC$ , mass  $M$  and density  $\rho$ . Let  $Cx$  be any line through  $C$  in the plane of the triangle



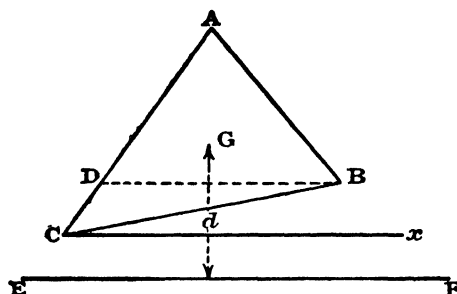


Fig. 94.

and let the line through B parallel to Cx cut AC at D. Let  $BD = b$  and let the perpendiculars from A and B to Cx be of lengths  $a$  and  $\beta$ .

The moment of inertia of the triangle ADB about a parallel axis through A is  $\frac{\rho b(a - \beta)^3}{4}$

and therefore about Cx the moment of inertia is

$$\frac{\rho b(a - \beta)}{2} \left[ \frac{(a - \beta)^2}{2} - \frac{4(a - \beta)^2}{9} + \frac{(a + 2\beta)^2}{9} \right].$$

The moment of inertia of the triangle BCD is  $\frac{\rho b \beta^3}{4}$ .

The moment of inertia of the triangle ABC is the sum of these two quantities and reduces to  $M \cdot \frac{a^3 + a\beta + \beta^3}{6}$ .

Consider a system of three particles each of mass  $\frac{M}{3}$  at the middle points of the sides of the triangle ABC. The centre of gravity coincides with that of the triangle and the moment of inertia about Cx is  $\frac{M}{3} \left[ \frac{a^2}{4} + \frac{(a + \beta)^2}{4} + \frac{\beta^2}{4} \right]$  i.e.  $M \frac{a^3 + a\beta + \beta^3}{6}$ .

Let EF be a line parallel to Cx at distance  $d$  from the centre of gravity, then the moment of inertia about EF of the triangle ABC and also of the system of three particles is

$$M \left[ \frac{a^3 + a\beta + \beta^3}{6} - \frac{(a + \beta)^3}{9} + d^2 \right].$$

Thus a triangle and a system of three particles, each one-third the mass of the triangle placed at the middle points of the sides, are equimomental about any axis parallel to the plane.

It follows that a parallelogram of mass  $M$  is equimomental to the system of five particles, four of  $\frac{M}{6}$  at the middle points of the sides and one of  $\frac{M}{3}$  at the point of intersection of the diagonals.

NOTE 1. The moments of inertia of triangles and quadrilaterals can more readily be obtained by using the result of this article than by methods involving integration.

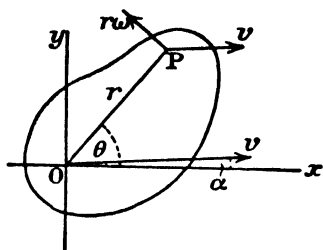
**52. Kinetic Energy of a Rotating Rigid Body.**

Fig. 95.

$\frac{1}{2} \omega^2 \Sigma m r^2$ , i.e.  $\frac{1}{2} \omega^2 M k^2$  where  $k$  is the radius of gyration of the body about the axis of rotation.

Suppose now that the axis through O passes through the centre of gravity of the body and that the latter is moving at any instant with velocity  $v$ , parallel to the plane  $xOy$  and making an angle  $\alpha$  with  $Ox$ . Let the plane through the axis and P make an angle  $\theta$  with the plane through the axis and  $Ox$ ; then, with the usual notation

$$\bar{x} = \frac{\Sigma m r \cos \theta}{\Sigma m} \text{ and } y = \frac{\Sigma m r \sin \theta}{\Sigma m}$$

so that, as the axis passes through the centre of gravity,

$$\Sigma m r \cos \theta = \Sigma m r \sin \theta = 0.$$

The kinetic energy of the particle is

$$\frac{1}{2} m \left( v^2 + r^2 \omega^2 + 2 v r \omega \cos \theta + \frac{\pi}{2} - \alpha \right),$$

consequently that of the whole body will be

$$\begin{aligned} & \frac{1}{2} v^2 \Sigma m + \frac{1}{2} \omega^2 \Sigma m r^2 - v \omega \Sigma m r \sin (\theta - \alpha). \\ &= \frac{1}{2} M v^2 + \frac{1}{2} \omega^2 M k^2 - v \omega \cos \alpha \cdot \Sigma m r \sin \theta + v \omega \sin \alpha \cdot \Sigma m r \cos \theta. \\ &= \frac{1}{2} M v^2 + \frac{1}{2} \omega^2 M k^2. \end{aligned}$$

Thus the kinetic energy of a rigid body having motions of translation and rotation is the sum of the kinetic energy due to translation supposing the whole mass to be concentrated at the centre of gravity and the kinetic energy due to rotation about an axis through its centre of gravity.

**Ex. VI.** The flywheel of a stationary engine weighs 2 cwt., and is in the form of a circular disc of radius 18 inches, and of uniform thickness and density. If the wheel revolve 120 times per minute, find the kinetic energy of the wheel due to its rotation.

$$\begin{aligned} \text{K.E. of flywheel} &= \frac{1}{2} \times 16\pi^2 \times 224 \times \frac{9}{8} \text{ ft. pdl.} \\ &= 630 \text{ ft. lb. (approx.).} \end{aligned}$$

**Ex. VII.** Find the kinetic energy of an anchor ring rolling without slipping at 10 miles an hour.

Mean radius of ring = 1 ft.; radius of section = 2"; specific gravity of material = 8.

$$\text{Mass } M \text{ of anchor ring} = 2\pi^2 \times \frac{1}{36} \times 1 \times 62.5 \times 8 \text{ lb.}$$

$$\text{K.E. due to translation} = \frac{1}{2}M \times \frac{44 \times 44}{9} \text{ ft. pdl.}$$

$$\text{K.E. due to rotation} = \frac{1}{2}M \times \left(\frac{44}{3} \times \frac{6}{7}\right)^2 \times \left(\frac{4 + \frac{1}{18}}{4}\right) \text{ ft. pdl.}$$

$$\begin{aligned} \therefore \text{ total K.E.} &= \frac{5000}{36} \times \frac{44 \times 44}{9} \times \left[1 + \frac{36}{49} \times \frac{49}{48}\right] \text{ ft. pdl.} \\ &= \frac{5000 \times 44 \times 44 \times 7}{36 \times 9 \times 4} \text{ ft. pdl.} \\ &= 1630 \text{ ft. lb. (approx.).} \end{aligned}$$

**NOTE 2.** It is proved in article 63 that the work done by the forces acting on a rigid body is equal to the kinetic energy (due to translation and rotation) produced.

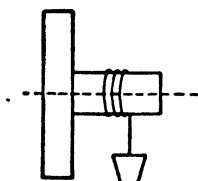


Fig. 96.

**Ex. VIII.** The figure shows a solid cylindrical wheel and its cylindrical axle. Diameter of wheel = 12 cm., diameter of axle = 3 cm., thickness of wheel = 2 cm., length of axle = 6 cm. A mass of 50 gm. which is hung by a light thread falls two metres in 10 sec. from rest. The wheel and axle are homogeneous and of the same material throughout. Find the specific gravity of the material of the wheel and axle.

$$\text{Work done after 10 sec.} = 10,000 \times 981 \text{ ergs.}$$

$$\text{Angular vel. of wheel and axle after 10 sec.} = \frac{40}{1.5} \text{ rad. per sec.}$$

Let  $s$  = specific gravity of the material, then

$$\frac{1}{2} \times \frac{80 \times 80}{9} \times \left[72\pi s \cdot \frac{36}{2} + \frac{27\pi}{2} s \cdot \frac{9}{8}\right] + 40000 = 9810000$$

$$\therefore 360\pi s [1280 + 15] + 40000 = 9810000$$

$$\begin{aligned} \therefore s &= \frac{9770000}{360 \times 1295\pi} \\ \therefore s &= 6.67 \text{ (nearly).} \end{aligned}$$

**Ex. IX.** A uniform disc, of radius  $a$ , is pivoted about a horizontal axis perpendicular to its plane, and passing through a point O, at distance  $x$  from the centre C. The disc is held with the diameter OC horizontal and is then let go. What is the value of its angular velocity at the instant when the diameter OC is vertical?

Show that for different points of suspension, the angular velocity is greatest when  $x = \frac{a}{\sqrt{2}}$ .

Let  $\omega$  be the angular velocity when OC is vertical,

then  $Mgx = \frac{1}{2}\omega^2 M \left\{ \frac{a^2}{2} + x^2 \right\}$  where M is the mass of the disc.

$$\therefore \omega = \left[ \frac{4gx}{a^2 + 2x^2} \right]^{\frac{1}{2}}$$

Now  $\omega$  is a maximum when  $\frac{4g}{\frac{a^2}{x} + 2x}$  is a maximum

i.e. when  $\frac{a^2}{x} + 2x$  is a minimum

i.e. when  $-\frac{a^2}{x^2} + 2 = 0$  i.e. when  $x = \frac{a}{\sqrt{2}}$ .

**Ex. X.** A sphere, radius  $a$ , rolls without slipping down a plane inclined at an angle  $\alpha$  to the horizontal. Find the acceleration when the sphere is (i.) a thin shell, (ii.) solid, (iii.) hollow, with an internal radius  $b$ .

Let A be the point of contact when the sphere is at rest and B, the point of contact after moving for time  $t$  from rest. Let  $AB = x$ ,  $M \equiv$  the mass of the sphere and  $k \equiv$  its radius of gyration about an axis through the centre.

$$\text{Then } Mgx \sin \alpha = \frac{1}{2} \cdot \frac{(dx)^2}{dt^2} M \{ k^2 + a^2 \}$$

$$\therefore g \sin \alpha \cdot \frac{dx}{dt} = \frac{k^2 + a^2}{a^2} \cdot \frac{dx}{dt} \cdot \frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2x}{dt^2} = \frac{a^2}{k^2 + a^2} \cdot g \sin \alpha.$$

For case (i.),  $k^2 = \frac{2}{3}a^2$ , so that the acceleration is  $\frac{3}{5}g \sin \alpha$ .

„ „ (ii.),  $k^2 = \frac{2}{5}a^2$ , „ „ „ „ „ „  $\frac{5}{7}g \sin \alpha$ .

„ „ (iii.),  $\frac{4}{3}\pi(a^3 - b^3) \cdot k^2 = \frac{8}{3}\pi \int_b^a x^4 dx = \frac{8}{15}\pi(a^5 - b^5)$

$\therefore k^2 = \frac{2}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}$ , so that the accel. is  $\frac{5a^2(a^3 - b^3)}{7a^5 - 5a^2b^3 - 2b^5} \cdot g \sin \alpha$ .

**53. Compound Pendulum.** A body can oscillate freely about a fixed horizontal axis: it is required to find the periodic time of a small oscillation.

The figure shows that vertical section of the body which passes through the centre of gravity G and which is at right angles to the axis which passes through O.

Let  $OG = h$ , let  $k$  be the radius of gyration of the body about a parallel axis through G and M the mass. If  $\theta$  be the

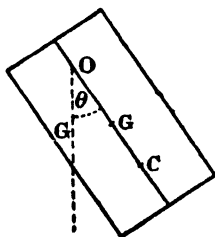


Fig. 97.

angular velocity of the body when OG is vertical, then when OG is inclined at an angle  $\theta$  to the vertical,

$$\frac{1}{2} \omega^2 \cdot M \cdot (k^2 + h^2) - \frac{1}{2} \cdot \left( \frac{d\theta}{dt} \right)^2 M (k^2 + h^2) = Mgh (1 - \cos \theta)$$

$$\therefore -\frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} \cdot (k^2 + h^2) = gh \sin \theta \cdot \frac{d\theta}{dt}.$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2 + h^2} \cdot \sin \theta.$$

For small oscillations, the equation of motion is

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2 + h^2} \cdot \theta.$$

Thus the motion is simple harmonic, the periodic time being

$$2\pi \sqrt{\frac{k^2 + h^2}{gh}} \text{ and the length of the simple equivalent pendulum being } h + \frac{k^2}{h}.$$

If a point C be taken on OG produced such that  $OC = h + \frac{k^2}{h}$ , O and C are known as the centres of suspension and oscillation respectively.

Suppose now the body to be suspended at C, then the time of oscillation is obtained by substituting  $\frac{k^2}{h}$  for  $h$  in the expression

$$2\pi \sqrt{\frac{k^2 + h^2}{gh}} \text{ and is therefore } 2\pi \sqrt{\frac{k^2 + \frac{k^4}{h^2}}{\frac{g}{h}}} \text{ i.e. } 2\pi \sqrt{\frac{k^2 + h^2}{gh}}.$$

Thus the centres of suspension and oscillation are convertible for if the body be suspended from either it will make small vibrations in the same time as a simple pendulum whose length is the distance between these centres.

Kater's Pendulum is so contrived that for any centre of suspension, O, the position of the centre of oscillation, C, can be experimentally determined and thus the value of  $g$

obtained from the formula  $T = 2\pi \sqrt{\frac{l}{g}}$ , where  $T$  is the periodic

time of the oscillation when suspended from either O or C and  $l$  is the distance OC.

**Ex. XI.** A disc, mass  $M$  and radius  $a$ , has a thin rigid rod, mass  $nM$  and length  $4a$ , attached normally to a point on its circumference. The compound pendulum thus formed is suspended from the other end of the rod : what is the length of the simple equivalent pendulum if it oscillates in the plane containing the disc ?

If  $C$  is the middle point of the rod,  $O$  the centre of the disc,  $A$  the upper end of the rod and  $G$ , the C.G. of the whole,

$$\text{then } CG = \frac{3aM}{M + nM} = \frac{3a}{1 + n} \quad \text{and} \quad GO = \frac{3an}{1 + n}.$$

Also, if  $k$  be the radius of gyration of the whole about an horizontal axis through  $G$  perpendicular to the plane containing the disc, then

$$M(1 + n)k^2 = M\left[\frac{a^2}{2} + \frac{9a^2n^2}{(1 + n)^2}\right] + nM\left[\frac{4a^2}{3} + \frac{9a^2}{(1 + n)^2}\right]$$

$$\therefore 6(1 + n)^2k^2 = a^2(3 + 65n + 8n^2).$$

The length of the simple equivalent pendulum is  $AG + \frac{k^2}{AG}$

$$\begin{aligned} &= \frac{a(5 + 2n)}{1 + n} + \frac{a(3 + 65n + 8n^2)}{6(1 + n)(5 + 2n)} \\ &= \frac{153 + 185n + 32n^2}{6(1 + n)(5 + 2n)} \cdot a = \frac{153 + 32n}{6(5 + 2n)} \cdot a. \end{aligned}$$

**Ex. XII.** A thin hollow cylinder, radius  $a$ , mass  $M$ , open at both ends, rests on a perfectly rough horizontal plane. Inside, along the lowest generating line, there is a thin rod also of mass  $M$ , fastened to the cylinder. If a slight displacement be made, find the periodic time of the oscillations which take place.

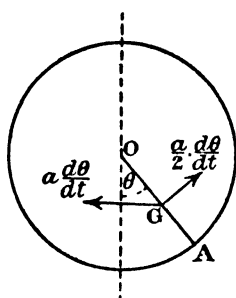


Fig. 98.

Let  $O$  be the centre of the cylinder and  $A$  the position of the C.G. of the rod, then  $G$  the middle point of  $OA$  is the C.G. of the system. If  $k$  be the radius of gyration of the system about an axis through  $G$  parallel to the axis of the cylinder, then

$$\begin{aligned} 2Mk^2 &= M \cdot \frac{a^2}{4} + M \left( a^2 + \frac{a^2}{4} \right) \\ \therefore k^2 &= \frac{3a^2}{4}. \end{aligned}$$

Suppose that, the displacement having been made, the system is at rest when  $OA$  makes an angle  $\alpha$  with the vertical : then when  $OA$  makes an angle  $\theta$  with the vertical, the angular velocity  $\frac{d\theta}{dt}$  is given by

$$\frac{1}{2} \left( \frac{d\theta}{dt} \right)^2 \cdot 2M \left[ \left( a^2 + \frac{a^2}{4} + a^2 \cos \pi - \theta \right) + k^2 \right] = 2Mg \cdot \frac{a}{2} (\cos \theta - \cos \alpha)$$

$$\text{i.e. } \left( \frac{d\theta}{dt} \right)^2 a(2 - \cos \theta) = g(\cos \theta - \cos \alpha) \quad \dots \quad \dots \quad (1)$$

Differentiating with respect to  $t$ , we have

$$2 \frac{d^2\theta}{dt^2} a(2 - \cos \theta) + \left(\frac{d\theta}{dt}\right)^2 a \sin \theta = -g \sin \theta. \dots \dots (2)$$

Now if the displacement be slight so that  $\alpha$  and  $\theta$  are small quantities of the 1st order, the R.H.S. of equation (1) becomes  $g \left[ \left(1 - \frac{\theta^2}{2} + \dots\right) - \left(1 - \frac{\alpha^2}{2} + \dots\right) \right]$  and thus  $\left(\frac{d\theta}{dt}\right)^2$  is a small quantity of the 2nd order. Neglecting small quantities of higher order than the first, equation (2) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{2a} \theta.$$

$\therefore$  the periodic time of small vibrations is  $2\pi \sqrt{\frac{2a}{g}}$ .

### Problems VII.

1. A thin trapezium has its parallel sides of length  $l_1$  and  $l_2$  and  $d$  is the distance between them. Find the moment of inertia about the side of length  $l_1$ .

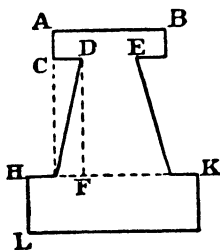


Fig. 99.

2. The figure shows the section of a girder:  $AB = DF = 4''$ ,  $AC = 1''$ ,  $HL = DE = 2''$ ,  $HK = 6''$ . Find the moment of inertia of the area of the section about an axis through its centroid parallel to  $AB$ .

3. A girder is formed of four equal rectangular plates  $5''$  wide and  $\frac{1}{4}''$  thick. In section they enclose a rectangle  $5''$  by  $3\frac{1}{4}''$ , the top and bottom plates overlapping the side plates by  $\frac{1}{4}''$  on each side. Find the moment of inertia of the area of the section about an axis in the plane of the section through the centre of gravity and parallel to the top and bottom plate.

4. Show that the moment of inertia of a solid elliptical cylinder about its axis is  $M \frac{a^2 + b^2}{4}$ .

5. Show that the moment of inertia of a rectangle whose sides are  $l, b$  about a diagonal is  $M \frac{l^2 b^2}{6(l^2 + b^2)}$ .

6. A right circular cone has mass  $M$  and height  $h$  and  $a$  is the radius of the base. Show that the moment of inertia about the axis is  $M \frac{3}{10} a^2$ ; that about a line through the vertex perpendicular to the axis is  $M \frac{3}{20} (4h^2 + a^2)$ ; and that about a line through the centre of gravity perpendicular to the axis is  $M \frac{3}{80} (h^2 + 4a^2)$ .

7. Show that the moment of inertia of a truncated cone about its axis of symmetry is  $M \frac{3}{10} \frac{r_1^5 - r_2^5}{r_1^3 - r_2^3}$ .

8. A cylindrical bar is 20 in. long and the diameter of its section is 1 in. Prove that  $h^2$  is greater than if the thickness had been neglected in the ratio 1.001875 : 1.

9. ABC is a triangular lamina of mass  $m$ , in which  $AB = AC$ , and AD is perpendicular to BC. Show that the lamina has the same moment of inertia about AD and BC as the system consisting of three particles each of mass  $\frac{1}{3}m$  situated at the middle points of the sides of the triangle.

10. A parabolic area is bounded by a double ordinate at a distance  $h$  from the vertex; show that its moment of inertia about an axis through its centroid perpendicular to the axis of the parabola and in the plane of the parabola is  $\frac{1.2}{17.5} Mh^2$  and that about the axis of the parabola is  $\frac{1}{4}Mah$ .

11. Show that the moment of inertia of a paraboloid of revolution about its axis is  $\frac{M}{3} y^2$  where  $y$  is the radius of its base.

12. Prove that the moment of inertia of an anchor ring whose cross section is a thin hollow circle is  $\frac{M}{2}(2R^2 + 3r^2)$ .

13. Masses of  $m_1$  and  $m_2$  ( $m_1 > m_2$ ) are connected by a light string which passes over a solid cylindrical pulley of mass  $M$  which is free to rotate about its centre. Find the acceleration of the masses  $m_1$  and  $m_2$  when the system is free to move.

14. Show that the moment of inertia of the area enclosed by the lemniscate  $r^2 = a^2 \cos 2\theta$  about a line through the pole perpendicular to its plane is  $M \frac{\pi a^2}{8}$ .

15. A cord 875 ft. long is wrapped round the axle of a heavy wheel and is pulled with a constant force of 75 lb., causing the wheel to rotate. When the cord is unwound and comes off, the wheel is found to be making 95.5 revolutions per minute. Calculate the moment of inertia of the wheel.

16. A top weighing 1 lb. and having a radius of gyration of 1 inch has a string 2 feet long coiled round it; it is set spinning by pulling this string with a uniform force of 1 lb. weight. Apply the principle of energy to find the angular velocity thus imparted and give the result in revolutions per minute.

17. A mass of 100 lb. is suspended by a cord which is wrapped round the rim of a fly-wheel of 4 ft. diameter. The mass is released from rest and falls a distance of 6 feet in 10 sec. Neglecting the friction of the bearings and the weight of the cord, find (i.) the ratio of the kinetic energy of the fly-wheel to that of the weight, and (ii.) the moment of inertia of the fly-wheel.



18. How many foot-pounds of energy are there in a uniform circular disc, mass 100 lb., diameter 3 feet, when it is rotating at a speed of 500 revolutions per minute? How many revolutions will the disc make before coming to rest if a frictional force equal to 10 lb. weight is applied tangentially at its rim?

19. A solid homogeneous cylinder rolls a certain distance down an inclined plane in 1 second; a cylindrical tube takes 1.1 seconds to cover the same distance. Find the ratio of the internal radius to the external radius.

20. Prove that about the diameter of a homogeneous sphere  
 $(\text{radius of gyration})^2 = 0.1 (\text{diameter})^2$ ;  
 and show how this may be verified experimentally by allowing the sphere to roll down a slit of uniform breadth in an inclined plane.

21. A uniform wire in the form of a circle of radius  $a$  swings in a vertical plane about a point A in the circumference. It starts from rest with the diameter AB horizontal. Find its angular velocity when AB is vertical.

22. A solid homogeneous sphere is attached to a fixed horizontal tangential bar, of negligible thickness, about which it can turn freely. The sphere, being at rest with its centre vertically over the bar, is slightly displaced and falls. Find the velocity with which the centre of the sphere passes under the bar, and the whole kinetic energy of the sphere at this moment.

23. A fly-wheel is movable in smooth bearings, about a horizontal axis, and is set in motion by a descending body of mass P which hangs from one end of a cord coiled round the axle of radius  $r$ . This body is found to descend  $h$  ft. in  $n$  secs. from rest; prove the moment of inertia of the wheel about its axis is  $Pr^2 \left( \frac{gn^2}{2h} - 1 \right)$ .

24. A thin ring of mass  $m$  has a particle of mass  $m$  fixed to its circumference. It rolls from rest down a plane inclined at angle  $\alpha$  to the horizon and initially the particle is in contact with the plane. What is the angular velocity and angular acceleration of the ring when it has made half a revolution? Obtain a definite integral giving the time taken to make the first complete revolution.

25. One half of a thin hollow cylinder, radius  $a$ , cut off by a plane through its axis is performing small oscillations on a perfectly rough horizontal table. Show that the time of a small oscillation

$$\text{is } 2\pi \sqrt{\frac{(\pi - 2)a}{g}}.$$

26. A hemisphere, radius  $a$ , performs small oscillations on a perfectly rough horizontal table. Show that if the hemisphere is

hollow and thin the time of a small oscillation is  $2\pi \sqrt{\frac{4a}{3g}}$  while if

solid and of uniform density the time is  $2\pi \sqrt{\frac{26a}{15g}}$ .

27. A cone, height  $h$  and radius  $a$  oscillates about a horizontal axis along a diameter of the base. Show that the length of the simple equivalent pendulum is  $\frac{3a^2 + 2h^2}{5h}$ .

28. A uniform rod, mass  $M$  and length  $l$  has a particle of mass  $m$  attached to one end. Find the length of the simple equivalent pendulum if it oscillate about a horizontal axis through the other end.

29. AB is a uniform rod of length  $l$  and mass  $m$  and at A a particle of mass  $m$  is attached. It oscillates about a horizontal axis through P where  $AP = x \left( x < \frac{l}{4} \right)$ . What is the length of the simple equivalent pendulum?

30. ABC is a triangle formed by three thin rods each of mass  $m$  and of length  $l$ . It oscillates about a horizontal axis through A: find the lengths of the simple equivalent pendulums when the oscillations are (i.) in the plane ABC, (ii.) at right angles to the plane ABC.

31. If the compound pendulum referred to in Ex. XI. oscillate at right angles to the plane containing the disc, find the periodic time.

32. A uniform rod of mass  $M$  and length  $2a$  can turn freely in a vertical plane about one end which is fixed. A cross-piece of mass  $2M$  and length  $a$  can be fixed at any point of the rod, with its centre on the rod, in the vertical plane in which the rod can move. Show that this cross-piece can be attached so that the length of the simple equivalent pendulum may have any value between  $a$  and  $\frac{19a}{10}$  and that there are two positions of the cross-piece corresponding to any value between  $a$  and  $\frac{3}{2}a$ .

33. A uniform rod of mass  $M$  has two small light rings fastened to its ends in such a manner that the rings can slide along a smooth fixed wire in the form of a vertical circle of radius  $a$  and centre O. The rod subtends an angle  $2\alpha$  at O, and its centre of gravity is denoted by G. The rod is disturbed from its position of stable equilibrium; show that when OG makes an angle  $\theta$  with the downward drawn vertical, the total energy of the system differs from the expression

$$\frac{1}{2}Ma^2(\cos^2\alpha + \frac{1}{3}\sin^2\alpha)\dot{\theta}^2 - Mga \cos \alpha \cos \theta$$

by a constant.

Deduce that if the rod is slightly disturbed from its position of stable equilibrium then G will describe a simple harmonic motion and determine the period.

## CHAPTER VIII

### EQUILIBRIUM OF STRINGS

#### 54. Light String on a Rough Curve.

APB is the curve, the string leaving the curve at A and B. Let  $\alpha, \beta$  be the angles the tangents at A and B make with a fixed line  $Ox$  and let  $T_1, T_2$  be the tensions at these points. Consider an element PQ of length  $\delta s$ ; let  $\theta, \theta + \delta\theta$

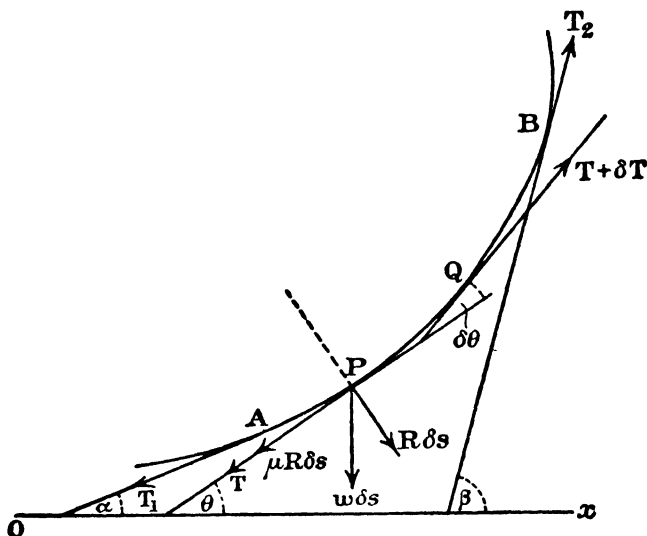


Fig. 100.

be the angles the tangents at these points to the curve make with  $Ox$  and let  $T, T + \delta T$  be the tensions respectively. Also let  $R$  be the normal pressure per unit length of the curve on the portion PQ of the string, and  $\mu$  the co-efficient of friction.

When the string is on the point of slipping in the direction from A to B, we have, resolving along the tangent for the element PQ,

$$(T + \delta T) \cos \delta\theta - T = \mu R \delta s \quad \dots \quad (1)$$

and resolving along the normal,

$$(T + \delta T) \sin \delta\theta = R \delta s. \quad \dots \quad (2)$$

Neglecting small quantities of higher order than the 1st, these equations become

$$\delta T = \mu R \delta s \quad \dots \quad (3)$$

$$\text{and} \quad T \delta\theta = R \delta s \quad \dots \quad (4)$$

$$\frac{\delta T}{T} = \mu \delta \theta.$$

Thus for the portion AB of the string, we have

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_a^\beta d\theta.$$

$$\therefore \log \frac{T_2}{T_1} = \mu(\beta - \alpha),$$

$$\therefore T_2 = T_1 \cdot e^{\mu(\beta - \alpha)}.$$

**55. Heavy String on a Curve.** Let  $w$  be the weight per unit length of the string, then referring to figure 100,  $Ox$  being horizontal, and resolving the forces on the element  $PQ$  along the tangent we have,

$$(T + \delta T) \cos \delta \theta - T - \mu R \delta s - w \delta s \sin \theta = 0 \quad (1)$$

and resolving along the normal,

$$(T + \delta T) \sin \delta \theta - R \delta s - w \delta s \cos \theta = 0. \quad \dots (2)$$

Neglecting small quantities of higher order than the 1st, these equations become,

$$\delta T - \mu R \delta s - w \delta s \sin \theta = 0 \quad \dots \dots (3)$$

$$\text{and} \quad T \delta \theta - R \delta s - w \delta s \cos \theta = 0. \quad \dots \dots (4)$$

When the equation to the curve is known,  $\delta s$  can be expressed in terms of  $\theta$  and  $\delta \theta$  and by the elimination of  $R$  from the equations (3) and (4), a differential equation is obtained

involving  $\frac{dT}{d\theta}$ ,  $T$  and  $\theta$ . The solution to this equation gives

the relation between  $T_1$ ,  $T_2$ ,  $\alpha$ ,  $\beta$  and  $w$ .

In the particular case when the curve is smooth and the chain is in equilibrium, equation (3) becomes

$$\delta T = w \delta s \sin \theta.$$

Suppose for the curve,  $\delta s = f(\theta) \delta \theta$

$$\text{then} \quad T_2 - T_1 = w \int_a^\beta f(\theta) \sin \theta d\theta.$$

$$\text{The tension } T \text{ at } P \text{ is given by } T - T_1 = w \int_a^\theta f(\theta) \sin \theta d\theta$$

and by substituting this value of  $T$  in equation (4), the reaction at any point on the chain can be determined.

**Ex. I.** A light string passes over a rough horizontal cylindrical beam and at one end is a weight of 1 cwt. It is found that a weight of 40 lb. when tied to the other end just prevents slipping. What would be the least weight necessary to prevent slipping if the string were wound an additional time round the beam ?

Let  $\mu$  be the co-efficient of friction, then

$$\frac{112}{40} = e^{\mu\pi} \quad \dots \quad \dots \quad \dots \quad (1)$$

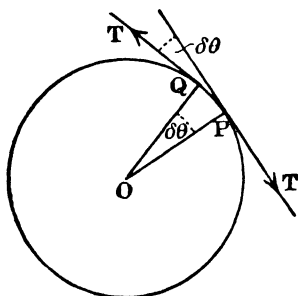
Let  $W$  lb. be the least weight required to support the weight of one cwt. when the angle is  $3\pi$ , then

$$\frac{112}{W} = e^{3\mu\pi}, \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\therefore \frac{W}{112} = \left(\frac{40}{112}\right)^3,$$

$$\therefore W = 5 \frac{5}{49} \text{ lb.}$$

**Ex. II.** A thin circular wire radius  $a$  ft. and mass  $M$  lb. rotates with angular velocity  $\omega$  in a horizontal plane about a fixed vertical axis through its centre. Find the tension of the wire.



Consider an element  $PQ$  of the wire subtending an angle  $\delta\theta$  at the centre  $O$ . Let  $T$  pdl. be the tension, then considering the motion of the portion  $PQ$  along  $OP$ , we have

$$T \sin \delta\theta = \frac{Ma\delta\theta}{2a\pi} \cdot a\omega^2.$$

If  $\delta\theta$  be small so that  $(\delta\theta)^2$  and higher powers can be neglected we have

$$T = \frac{M}{2\pi} a\omega^2 \text{ pdl.}$$

Fig. 101.

**Ex. III.** A heavy uniform chain, weight  $w$  lbs. per unit length rests on a smooth circular arc, radius  $a$ , in a vertical plane. The chain occupies a quadrant with one end at the highest point and is kept in equilibrium by a force  $F$  at the upper end, applied horizontally.

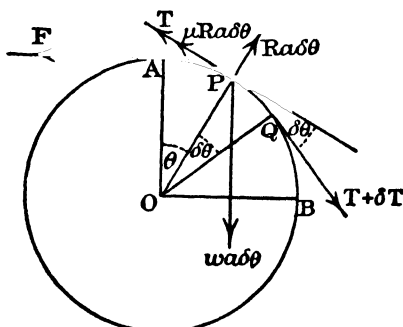


Fig. 102.

tally. Find the magnitude of  $F$ . If the arc be rough and the chain be on the point of slipping when  $F$  is zero, prove that the co-efficient of friction  $\mu$  satisfies the

$$\mu\pi \quad \text{equation } (1 - \mu^2)e^{\mu\pi} = 2\mu.$$

With the notation of Article 54, resolving the forces on the element  $PQ$  along the tangent and the normal at  $P$ , and neglecting small quantities of higher order than the first, we

have  $\delta T + wa\delta\theta \sin \theta - \mu Ra\delta\theta = 0$  (1)

and  $T\delta\theta + wa\delta\theta \cos \theta - Ra\delta\theta = 0$ . (2)

In the first case,  $\mu = 0$  and equation (1) becomes

$$\delta T = -wa \sin \theta \cdot \delta\theta,$$

$$\therefore \int_F^0 dT = -wa \int_0^{\frac{\pi}{2}} \sin \theta d\theta,$$

$$\therefore F = wa.$$

In the second case, eliminating R from equations (1) and (2) we have

$$\frac{dT}{d\theta} - \mu T = wa(\mu \cos \theta - \sin \theta). \quad \dots \dots \dots (3)$$

Multiplying both sides by  $e^{-\mu\theta}$  and integrating with respect to  $\theta$  we have

$$T \cdot e^{-\mu\theta} = wa\mu \int e^{-\mu\theta} \cos \theta d\theta - wa \int e^{-\mu\theta} \sin \theta d\theta + C, \quad (4)$$

$$\begin{aligned} \text{Now } \int e^{-\mu\theta} \sin \theta d\theta &= -\frac{1}{\mu} e^{-\mu\theta} \sin \theta + \frac{1}{\mu} \int e^{-\mu\theta} \cos \theta d\theta \\ &= -\frac{1}{\mu} e^{-\mu\theta} \sin \theta + \frac{1}{\mu} \left[ -\frac{1}{\mu} e^{-\mu\theta} \cos \theta - \frac{1}{\mu} \int e^{-\mu\theta} \sin \theta d\theta \right], \end{aligned}$$

$$\therefore \int e^{-\mu\theta} \sin \theta d\theta = -\frac{e^{-\mu\theta}}{1 + \mu^2} (\cos \theta + \mu \sin \theta).$$

$$\text{Similarly } \int e^{-\mu\theta} \cos \theta d\theta = \frac{e^{-\mu\theta}}{1 + \mu^2} (\sin \theta - \mu \cos \theta).$$

Equation (4) therefore becomes

$$T \cdot e^{-\mu\theta} = wa \frac{e^{-\mu\theta}}{1 + \mu^2} [\cos \theta + 2\mu \sin \theta - \mu^2 \cos \theta] + C.$$

$$\text{Now when } \theta = 0, T = 0, \therefore 0 = \frac{wa}{1 + \mu^2} [1 - \mu^2] + C \dots (5)$$

$$\text{and when } \theta = \frac{\pi}{2}, T = 0 \therefore 0 = \frac{wa}{1 + \mu^2} e^{-\frac{\mu\pi}{2}} [2\mu] + C \dots (6)$$

Eliminating C from equations (5) and (6), we have

$$(1 - \mu^2) e^{\frac{\mu\pi}{2}} = 2\mu.$$

**56. Uniform String Under the Action of Gravity.** We shall assume that the strings and chains discussed in this chapter offer no resistance to bending at any point. They are then said to be perfectly flexible, and the action across any section of the string is then tangential to the string at that section. We shall also assume the section to be so small that the form of the string is that of a smooth curve. It should be noted that if the string or chain had rigidity of shape, then the resultant force across any section would be a force and a couple.

A chain, whose links are small compared with the radius of curvature, or even a wire rope if the radius of curvature be very large approximates to these ideal conditions.

We shall only consider one or two simple cases in which the curve is a *plane* curve.

The curve which a heavy chain of uniform density assumes, when it is suspended from its ends, is called the **common** or **uniform catenary**. The equation and properties of this curve will now be determined.

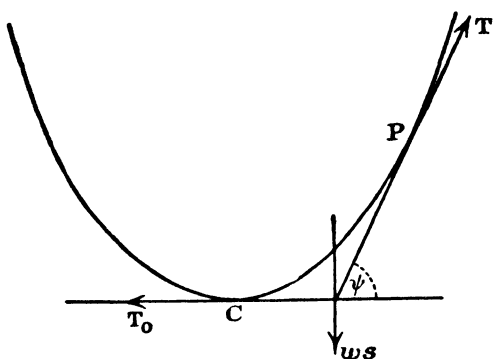


Fig. 103.

The simplest procedure is to write down the equations of equilibrium of a finite portion of the chain measured from the lowest point of that chain. Let C be the lowest point and CP the finite portion considered, of length  $s$ . Then the portion CP is in equilibrium under the action of  $T$ , the tension at P, making an angle  $\psi$ , with the horizontal,  $T_0$ , the horizontal tension at C and  $ws$ , the weight of CP, where  $w$  = weight of chain per unit length.

For equilibrium the lines of action of these forces must be concurrent.

Resolving horizontally and vertically, we have

$$T \cos \psi = T_0 \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad (1)$$

$$T \sin \psi = ws \quad \therefore \quad \therefore \quad \therefore \quad \therefore \quad (2)$$

Dividing (2) by (1)  $\tan \psi = \frac{ws}{T_0}.$

It is convenient to write the value  $T_0$  in the form  $wc$ , i.e. the tension of the string at its lowest point is the weight of a length  $c$  of the chain; then

$$\tan \psi = \frac{ws}{wc} = \frac{s}{c} \quad \dots \quad \therefore \quad \dots \quad (3)$$

$$\text{or} \quad s = c \tan \psi \quad \dots \quad \dots \quad (4)$$

Equation (4) is the intrinsic equation of the curve in which the chain hangs,  $s$  being measured, as already stated, from the lowest point at the curve. It is usual and also necessary to derive from this equation the cartesian equation of the curve.

Take as axes of  $x$  and  $y$ , horizontal and vertical lines respectively.

$$\text{Then} \quad \frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi \quad \dots \quad \dots \quad (5)$$

$$\text{and, by differentiating eqn. (4)} \quad \frac{ds}{d\psi} = c \sec^2 \psi. \quad \dots \quad (6)$$

$$\text{Now} \quad \frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi} = c \cos \psi : \sec^2 \psi = c \sec \psi \quad \dots \quad (7)$$

$$\text{and} \quad \frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi} = c \sin \psi \sec^2 \psi \quad \dots \quad \dots \quad (8)$$

Integrating eqns. (7) and (8) we have

$$x = c \log (\sec \psi + \tan \psi) \quad \dots \quad \dots \quad (9)$$

$$y = c \sec \psi \quad \dots \quad \dots \quad (10)$$

the constants of integration vanishing if the origin is taken at a distance  $c$  below the lowest point of the curve.

By eliminating  $\psi$  between equations (9) and (10) the cartesian equation of the curve is obtained.

Eqn. (9) may be written

$$e^{\frac{x}{c}} = \sec \psi + \tan \psi \quad \dots \quad \dots \quad (11)$$

$$\text{and since} \quad 1 = \sec^2 \psi - \tan^2 \psi$$

we have, by division

$$e^{-\frac{x}{c}} = \sec \psi - \tan \psi \quad \dots \quad \dots \quad (12)$$

Adding (11) and (12)

$$\begin{aligned} e^{\frac{x}{c}} + e^{-\frac{x}{c}} &= 2 \sec \psi \\ &= \frac{2y}{c}, \text{ from eqn. (10)} \end{aligned}$$

$$\therefore \quad y = \frac{c}{2} \left( e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right) = c \cosh \frac{x}{c} \quad \dots \quad \dots \quad (13)$$

which is the cartesian equation required.

Two equations connecting  $y$  and  $s$ , and  $x$  and  $s$  may also be obtained.



From (10)  $y^2 = c^2 \sec^2 \psi = c^2 (1 + \tan^2 \psi) = c^2 \left(1 + \frac{s^2}{c^2}\right)$  by (4)

$$\therefore y^2 = s^2 + c^2 \quad \dots \quad (14)$$

From (11) and (12), by subtraction

$$e^{\frac{s}{c}} - e^{-\frac{s}{c}} = 2 \tan \psi = \frac{2s}{c} \text{ from (4).}$$

$$\text{Hence} \quad s = \frac{c}{2} \left( e^{\frac{s}{c}} - e^{-\frac{s}{c}} \right) = c \sinh \frac{x}{c} \quad \dots \quad (15)$$

The straight lines which were taken as the axes of  $x$  and  $y$  are called respectively the **directrix** and the **axis** of the catenary, and the lowest point C is called the **vertex**. The constant " $c$ " is called the **parameter** of the catenary.

### 57. Properties of the Catenary.

Since  $T \cos \psi = T_0 = wc$

and  $y = c \sec \psi$

$$\therefore T = wc \sec \psi = wy \dots \quad (1)$$

Thus the tension at any point is equal to the weight of a portion of string the length of which is the distance of that point from the directrix. It therefore follows that if a chain be hung over two smooth pegs so that there is a portion at each peg hanging downwards and reaching to the directrix, the chain will remain in equilibrium.

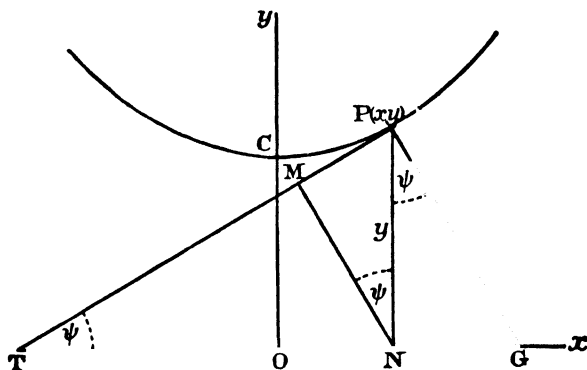


Fig. 104.

In the figure, PT is the tangent at P meeting the directrix at T.

PN is the ordinate and PG the normal at P. NM is the perpendicular from N on the tangent PT.

We have

$$\begin{aligned} NM &= y \cos \psi \\ &= c \sec \psi \cdot \cos \psi = c \end{aligned}$$

and

$$PM = c \tan \psi = s.$$

Again the radius of curvature ( $\rho$ ) at P is  $\frac{ds}{d\psi}$ .

But

$$s = c \tan \psi.$$

$$\therefore \rho = c \sec^2 \psi.$$

$$\text{Now } PG = y \sec \psi = c \sec^2 \psi \quad \therefore PG = \rho.$$

From the triangle PNM we also get

$$y^2 = s^2 + c^2.$$

Note also that for the vertex C,  $\rho = CO = c$ , so that if the chain has a large radius of curvature,  $c$  is also very large.

But  $T_0 = wc \therefore T_0$  is large. Hence when the curvature of the chain is small, the tension is large and *vice versa*.

Again, when the curvature is small, the horizontal distance between C and any point P on the curve is approximately equal to the length  $s$  of the chain between C and P.

**Ex. IV.** If a rope of given length  $2l$  is hung from two supports at the same level, prove that  $\sigma^2 + 2c\sigma = l^2$ , where  $\sigma$  is the depth of the lowest point below the level of the supports, and find an exact equation giving the tension at the lowest point in terms of  $l$ ,  $\sigma$ ,  $w$ ,  $w$  being the weight of unit length of the rope. Give the ratio  $c/l$  to three significant figures when  $\sigma = l/1000$ .

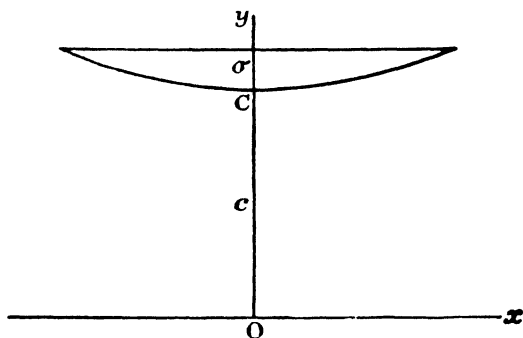


Fig. 105.

At the point of support

$$y = c + \sigma \text{ and } s = l.$$

But

$$y^2 = s^2 + c^2,$$

$\therefore$

$$(c + \sigma)^2 = l^2 + c^2$$

from which

$$l^2 = 2c\sigma + \sigma^2. \quad \dots \quad \dots \quad \dots \quad (1)$$

Hence

$$c = \frac{l^2 - \sigma^2}{2\sigma} \quad \dots \quad \dots \quad \dots \quad (2)$$

and  $T_o = wc = w \frac{l^2 - \sigma^2}{2\sigma}. \quad \dots \quad \dots \quad (3)$

From (2),  $c = \frac{l^2 \left(1 - \frac{\sigma^2}{l^2}\right)}{2\sigma} \therefore \frac{c}{l} = \frac{1 - \frac{\sigma^2}{l^2}}{2 \frac{\sigma}{l}}.$

When  $\sigma = \frac{1}{10^3}$ ,  $\frac{c}{l} = \frac{1 - \frac{1}{10^6}}{2 \frac{1}{10^3}} = \frac{10^6 - 1}{2 \times 10^3}$

= 500, correct to 3 sig. figures.

**Ex. V.** A uniform chain of length  $2l$ , is suspended at its ends which are on the same horizontal level. The distances apart ( $2a$ ) of the ends is such that the lowest point of the chain is a distance  $a$  vertically below the ends. Prove that, if  $c$  be the distance of the lowest point from the directrix of the catenary, then

$$c = \frac{l^2 - a^2}{2a}, \quad \tanh \frac{a}{c} = \frac{2al}{l^2 + a^2} \quad \text{and} \quad \frac{2a^2}{l^2 - a^2} = \log_e \frac{l+a}{l-a}.$$

Calculate in terms of  $a$  and  $l$  the distance of the centre of gravity of the chain below the ends.

As in the previous example,

$$\therefore \quad \begin{aligned} y^2 &= s^2 + c^2, \\ (c+a)^2 &= l^2 + c^2, \end{aligned}$$

from which  $c = \frac{l^2 - a^2}{2a}. \quad \dots \quad \dots \quad (1)$

Now  $l = c \sinh \frac{a}{c}, \quad \text{and} \quad a + c = c \cosh \frac{a}{c},$

$$\begin{aligned} \therefore \quad \tanh \frac{a}{c} &= \frac{l}{a+c} = \frac{l}{a + \frac{l^2 - a^2}{2a}}, \quad \text{from (1),} \\ &= \frac{2al}{l^2 + a^2}. \quad \dots \quad \dots \quad (2) \end{aligned}$$

Again,  $\tanh \frac{a}{c} = \frac{e^{\frac{a}{c}} - e^{-\frac{a}{c}}}{e^{\frac{a}{c}} + e^{-\frac{a}{c}}} = \frac{e^{\frac{2a}{c}} - 1}{e^{\frac{2a}{c}} + 1} = \frac{l}{a+c},$

$$\therefore \quad e^{\frac{2a}{c}}(a+c-l) = a+c+l;$$

from which  $e^{\frac{2a}{c}} = \frac{a+l + \frac{l^2 - a^2}{2a}}{a-l + \frac{l^2 - a^2}{2a}} = \frac{(l+a)(2a+l-a)}{(l-a)(l+a-2a)}$

$$= \frac{(l+a)^2}{(l-a)^2},$$

and therefore,  $\frac{a}{c} = \frac{l+a}{l-a},$

or  $\log_e \frac{l+a}{l-a} = \frac{a}{c} = \frac{2a^2}{l^2-a^2}$  from (1). ... (3)

If  $\bar{y}$  is the distance of the centre of gravity from the directrix, then

$$\bar{y} = \frac{\int wy ds}{\int w ds} = \frac{\int_0^l \sqrt{s^2 + c^2} ds}{\int_0^l ds}$$

since the catenary is symmetrical about the  $y$  axis.

$$\begin{aligned} \therefore \bar{y} &= \frac{1}{l} \left[ \frac{s}{2} \sqrt{s^2 + c^2} + \frac{c^2}{2} \log_e (s + \sqrt{s^2 + c^2}) \right]_0^l \\ &= \frac{1}{l} \left[ \frac{l}{2} \sqrt{l^2 + c^2} + \frac{c^2}{2} \log_e \frac{l + \sqrt{l^2 + c^2}}{c} \right]. \end{aligned}$$

But  $\sqrt{l^2 + c^2}$  = height above directrix at point of support.

$$= a + c = a + \frac{l^2 - a^2}{2a} = \frac{l^2 + a^2}{2a}.$$

$$\begin{aligned} \therefore \bar{y} &= \frac{1}{l} \left[ \frac{l}{2} \cdot \frac{l^2 + a^2}{2a} + \frac{(l^2 - a^2)^2}{8a^2} \log_e \frac{l + \frac{l^2 + a^2}{2a}}{\frac{l^2 - a^2}{2a}} \right] \\ &= \frac{l^2 + a^2}{4a} + \frac{(l^2 - a^2)^2}{8a^2 l} \log_e \frac{l+a}{l-a} \\ &= \frac{l^2 + a^2}{4a} + \frac{l^2 - a^2}{4l}, \quad \text{since } \log_e \frac{l+a}{l-a} = \frac{2a^2}{l^2 - a^2} \end{aligned}$$

Hence, if  $y'$  is the distance of the centre of gravity below the supports,

$$\begin{aligned} y' &= a + c - \bar{y} \\ &= \frac{l^2 + a^2}{4a} - \frac{l^2 - a^2}{4l}. \end{aligned}$$

**Ex. VI.** A uniform wire has a horizontal span of 108 ft. and the sag in the middle is 27 ft. Find the length of the wire.

If  $c$  is the parameter, then

$$27 + c = c \cosh \frac{54}{c}. \quad \dots \dots \dots (1)$$

and, if  $2l$  is the length of the wire,

$$l = c \sinh \frac{54}{c}. \quad \dots \dots \dots (2)$$

From eq. (1)  $c$  can be determined, and then eq. (2) gives the length of the wire.

It is important to notice that equation (1) can only be solved graphically.

Thus, putting  $\frac{54}{c} = X$ , we have

$$1 + \frac{1}{2}X = \cosh X.$$

If we plot the curve  $y = \cosh X$  i.e.,  $y = \frac{1}{2}(e^X + e^{-X})$  and also the straight line  $y = 1 + \frac{1}{2}X$ , then the abscissae of their

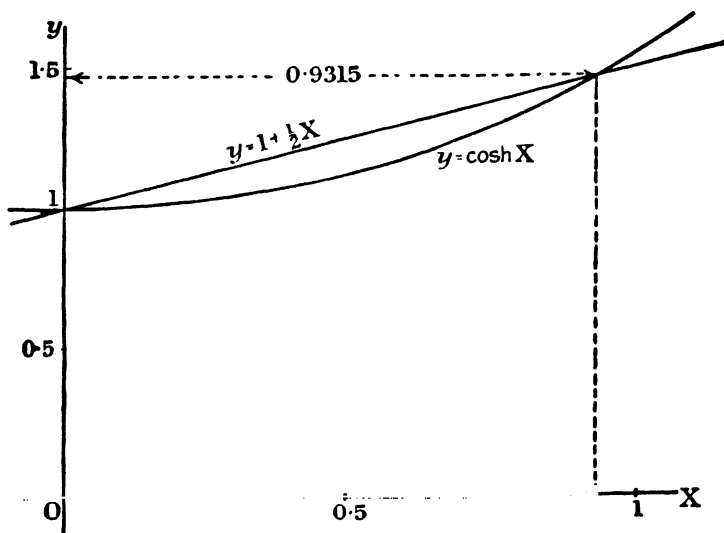


Fig. 106.

points of intersection will be the values of  $X$  satisfying the equation.

This method of solution gives to  $X$  the values  
0 and 0.9315 (approximately).

When  $X = 0$   $c$  is infinitely great, i.e., the wire would then be straight, which is contrary to hypothesis.

Hence  $X = 0.9315$

or 
$$c = \frac{54}{0.9315} = 57.97 \text{ feet.}$$

Substituting the above value of  $c$  in equation (2) we have

$$l = 57.97 \sinh .9315 = 57.97 \times 1.072 \text{ ft.} \\ = 62.155 \text{ ft. approximately.}$$

$$\text{length of wire} = 124.3 \text{ ft.}$$

NOTE 1. From consideration of the figure it is seen that if the slope of the wire is small, great accuracy is not possible. But the slope of the wire is given by the co-efficient of  $X$  which in the case considered is  $\frac{27}{54}$ , that is

$$\frac{\text{sag}}{\frac{1}{2} \text{ distance between the supports}}.$$

If then the length of the wire only slightly exceeds the span,  $c$  cannot be accurately determined.

NOTE 2. If the length  $2l$  of the wire, and its span  $2a$  were given, then the equation to determine the parameter  $c$  would be

$$l = c \sinh \frac{a}{c}. \quad (i.)$$

Putting  $\frac{a}{c} = X$ , we have

$$\frac{l}{a} X = \sinh X.$$

If we draw the line  $y = \frac{l}{a} X$  and the curve  $y = \sinh X$ , the abscissae of their points of intersection will give the required values of  $X$ , and therefore of  $c$ .

From the figure we see that there are two equal and opposite values of  $X$ , the negative value of which is inadmissible, since  $c$  is positive. As above, great accuracy is not possible if  $\frac{l}{a}$  is small, that is when the wire is nearly straight.

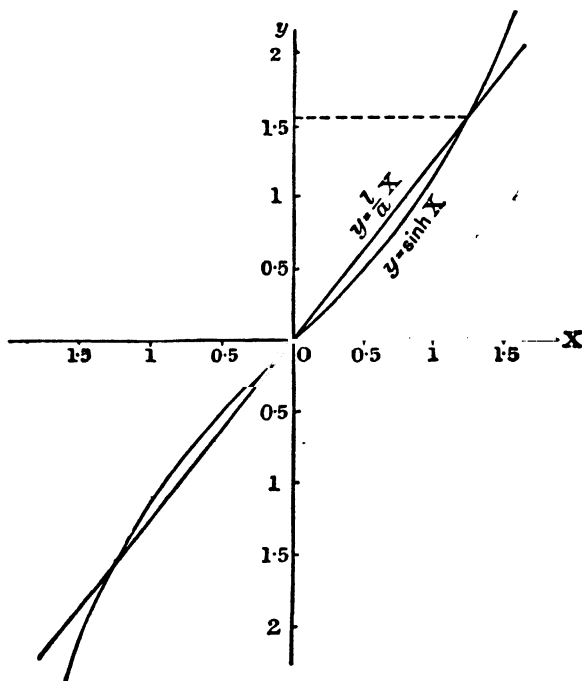


Fig. 107.

**58. Wire nearly horizontal.** This case requires special consideration. Let  $2a$  be the span, and  $2l$  the length of the wire.

Then 
$$l = \frac{c}{2} \left( e^{\frac{a}{c}} - e^{-\frac{a}{c}} \right). \quad \dots \quad (1)$$

When the wire is nearly straight,  $c$  is very large and  $\frac{a}{c}$  is small.

Expanding the exponentials in equation (1), we have

$$l = \frac{c}{2} \left( 1 + \frac{a}{c} + \frac{a^2}{2c^2} + \frac{a^3}{6c^3} - 1 + \frac{a}{c} - \frac{a^2}{2c^2} + \frac{a^3}{6c^3} \right),$$

retaining only the first three powers of  $\frac{a}{c}$ .

$$\therefore l = a + \frac{a^3}{6c^2}.$$

or 
$$c^2 = \frac{a^3}{6(l-a)}. \quad \dots \quad (2)$$

Again the tension at any point is  $wy$ , and as  $y$  is nearly the same throughout the wire, it is approximately constant and equal to  $wc$ .

If then  $W$  is the total weight of the wire,

$$2lw = W$$

and 
$$T = wc = \frac{W}{2l} \sqrt{\frac{a^3}{6(l-a)}} \quad \dots \quad (3)$$

We can also obtain an approximate value for the sag at the middle of the wire.

The ordinate at the point of support is given by

$$\begin{aligned} y &= \frac{c}{2} \left( e^{\frac{a}{c}} + e^{-\frac{a}{c}} \right) \\ &= \frac{c}{2} \left( 1 + \frac{a}{c} + \frac{a^2}{2c^2} + \frac{a^3}{6c^3} + \dots + 1 - \frac{a}{c} + \frac{a^2}{2c^2} - \frac{a^3}{6c^3} + \dots \right) \\ &= c \left( 1 + \frac{a^2}{2c^2} \right) \text{ neglecting higher powers of } \frac{a}{c} \text{ than the third} \quad (4) \end{aligned}$$

Hence if  $\sigma$  is the sag,

$$\sigma = y - c = \frac{a^2}{2c}. \quad \dots \quad (5)$$

Hence using eqn. (2),  $\sigma = \frac{1}{2} \sqrt{6a(l-a)}. \quad \dots \quad (6)$

NOTE 3. If  $x/c$  is small

$$y = \frac{c}{2} \left( e^{x/c} + e^{-x/c} \right) = \frac{c}{2} \left( 2 + \frac{x^2}{c^2} + \frac{x^4}{12c^4} + \dots \right)$$

neglecting higher powers of  $\frac{x}{c}$  than the third

$$= c + \frac{x^2}{2c}, \text{ which is the equation of a parabola.}$$

Hence, near the vertex, or if the wire is nearly straight, the form of the curve approximates to that of the parabola.

**Ex. VII.** A weight  $W$  is suspended from a fixed point by a uniform string of length  $l$ , and weight  $w$  per unit of length. It is drawn aside by a horizontal force  $P$ . Show that, in the position of equilibrium, the distance of  $W$  from the vertical through the fixed point is

$$\frac{P}{w} \left\{ \sinh^{-1} \left( \frac{W + lw}{P} \right) - \sinh^{-1} \frac{W}{P} \right\}.$$

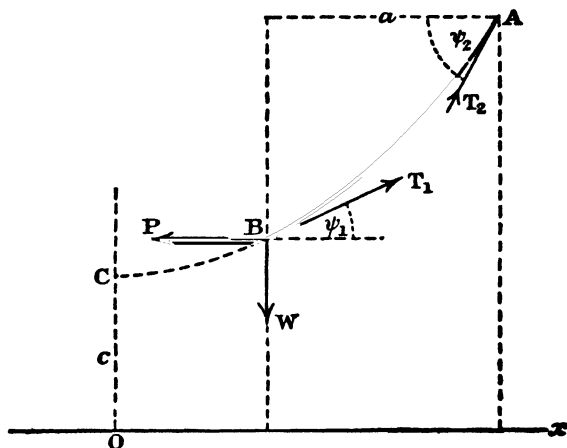


Fig. 108.

Let  $A$  be the fixed point, and  $AB$  the position of the string when equilibrium is attained.

Let  $C$  be the lowest point of the catenary of which  $AB$  is a part, and  $c$  the parameter of the catenary.

Let the distance of  $W$  from the vertical through  $A$  be  $a$ .

Let  $x_1$  be the abscissa of  $A$ ,  $x_2$  the abscissa of  $B$ ,

So that  $a = x_2 - x_1$ .

For any point on the catenary, with the usual axes,

$$y = c \cosh \frac{x}{c}.$$

Differentiating, 
$$\frac{dy}{dx} = \sinh \frac{x}{c}.$$



But  $\frac{dy}{dx} = \tan \psi$ , where  $\psi$  is the inclination of the tangent at the point under consideration.

$$\therefore \sinh \frac{x}{c} = \tan \psi. \quad \dots \quad \dots \quad \dots \quad (1)$$

Let  $\psi_1$  be the inclination at B,  $\psi_2$  the inclination at A and  $T_1, T_2$  the tensions at B and A respectively.

For the equilibrium of B,

$$\begin{aligned} T_1 \cos \psi_1 &= P, \\ T_1 \sin \psi_1 &= W. \end{aligned}$$

$$\therefore \tan \psi_1 = \frac{W}{P}.$$

$$\therefore \sinh \frac{x_1}{c} = \frac{W}{P}, \text{ from (1)}$$

$$\text{or} \quad x_1 = c \sinh^{-1} \frac{W}{P}. \quad \dots \quad (2)$$

For the equilibrium of AB we have

$$\begin{aligned} T_2 \cos \psi_2 &= P, \\ T_2 \sin \psi_2 &= W + wl. \end{aligned}$$

$$\therefore \tan \psi_2 = \frac{W + wl}{P},$$

$$\text{or} \quad \sinh \frac{x_2}{c} = \frac{W + wl}{P},$$

$$\text{so that} \quad x_2 = c \sinh^{-1} \frac{W + wl}{P}. \quad \dots \quad \dots \quad (3)$$

But the horizontal tension is constant and equal to  $wc$ .

$$\therefore P = wc \text{ i.e. } c = \frac{P}{w}. \quad \dots \quad \dots \quad (4)$$

Hence from (2), (3) and (4)

$$a = x_2 - x_1 = \frac{P}{w} \left\{ \sinh^{-1} \left( \frac{W + wl}{P} \right) - \sinh^{-1} \frac{W}{P} \right\}.$$

**Ex. VIII.** A kite is flown with 600 feet of string from the hand to the kite, and a spring balance held in the hand shows a pull equal to the weight of 100 feet of the string inclined at an angle of  $30^\circ$  to the horizontal; find the vertical height of the kite above the hand.

Referring to Fig. 108, let AB be the string, and  $w$  its weight in lb. per foot.

Let the ordinates of B, A be  $y_1, y_2$  respectively.

The forces acting on AB are  $T_1, T_2$  and its weight  $600w$ .

Resolving horizontally and vertically, we obtain

$$T_2 \cos \psi_2 = T_1 \cos \psi_1 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$T_2 \sin \psi_2 = T_1 \sin \psi_1 + 600w. \quad \dots \quad \dots \quad \dots \quad (2)$$

But  $T_1 = 100w$  and  $\psi_1 = 30^\circ$

$$\therefore T_1 \cos \psi_1 = 100w \cdot \frac{\sqrt{3}}{2} = 50\sqrt{3} \cdot w.$$

$$\text{and } T_2 \sin \psi_2 = 50w + 600w = 650w.$$

Squaring and adding

$$T_2^2 = 430000w^2$$

i.e.

$$T_2 = w \times 100 \sqrt{43} \text{ lb. wt.}$$

$$\text{But } T_1 = wy_1 \therefore 100w = wy_1 \text{ or } y_1 = 100 \text{ ft.}$$

$$\text{and } T_2 = wy_2 \therefore 100 \sqrt{43} \cdot w = wy_2 \text{ i.e. } y_2 = 100 \sqrt{43} \text{ ft.}$$

$$\therefore \text{vertical height required} = y_2 - y_1 \\ = 100 (\sqrt{43} - 1) \text{ ft.} = 556 \text{ ft. approximately.}$$

**Ex. IX.** A heavy uniform string of length  $l$  hangs from two points in the same horizontal line whose distance apart is  $d$ , where  $(l-d)/d$  is small. A weight equal to the weight of a length  $2s'$  of the string is attached to the middle point of the string; prove that the parameter  $c_1$  of the two equal catenaries in which the string hangs after the weight is attached is given by  $c_1 = c(1 + 3s'/d)$ , and that the sag is increased in the ratio  $(d + s')/d$ , where  $c$  is the parameter of the catenary in which the string hangs before the weight is attached, and it is assumed that  $(s'/d)^2$  is negligible.

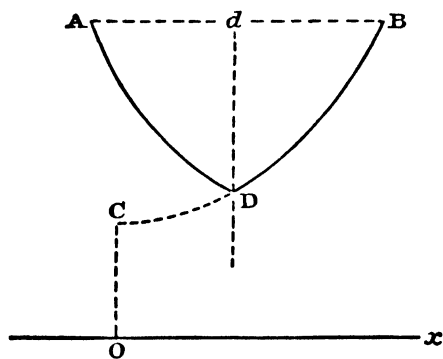


Fig. 109.

Let A and B be the two points of support, and D the point of suspension of the weight. If  $w$  is the weight of unit length of the string, then, weight suspended  $= 2s'w$ .

Let C be the vertex of the catenary of which BD is a part, and  $x$  the abscissa of D, measured from the vertical through C.

Now

$$\frac{l}{2} = BD = CB - CD$$

$$\therefore \frac{l}{2} = \frac{c_1}{2} \left( e^{\frac{x}{c_1} + \frac{d}{2c_1}} - e^{-\frac{x}{c_1} - \frac{d}{2c_1}} \right) - \frac{c_1}{2} \left( e^{\frac{x}{c_1}} - e^{-\frac{x}{c_1}} \right) \dots \quad (1)$$

Also  $2s'w =$  twice the vertical tension at D

$$= 2w \frac{c_1}{2} \left( e^{\frac{x}{c_1}} - e^{-\frac{x}{c_1}} \right) \dots \dots \dots (2)$$

$$\therefore s' = \frac{c_1}{2} \left( e^{\frac{x}{c_1}} - e^{-\frac{x}{c_1}} \right) = \frac{c_1}{2} \left( 1 + \frac{x}{c_1} - 1 + \frac{x}{c_1} \dots \right)$$

$$= x, \text{ approximately } \quad \text{since } \frac{l-d}{d} \text{ is small.} \quad \dots \quad (3)$$

$$\begin{aligned} \text{Now } \frac{l}{2} &= \frac{c}{2} \left( e^{\frac{d}{2c}} - e^{-\frac{d}{2c}} \right) \\ &= \frac{c}{2} \left( 1 + \frac{d}{2c} + \frac{d^3}{8c^3} + \frac{d^5}{48c^5} \dots - 1 + \frac{d}{2c} - \frac{d^3}{8c^3} + \frac{d^5}{48c^5} \dots \right) \\ &= \frac{d}{2} + \frac{d^5}{48c^4} \text{ approximately } \dots \quad (4) \end{aligned}$$

But, from (1)

$$\begin{aligned} \frac{l}{2} &= \frac{c_1}{2} \left( 1 + \frac{x}{c_1} + \frac{d}{2c_1} - 1 + \frac{x}{c_1} + \frac{d}{2c_1} + \frac{2 \left( \frac{x}{c_1} + \frac{d}{2c_1} \right)^3}{6} + \dots \right) - s' \\ &= x + \frac{d}{2} + \frac{1}{6c_1^3} \left( x + \frac{d}{2} \right)^3 - s' \\ &= \frac{d}{2} + \frac{1}{6c_1^3} \left( x + \frac{d}{2} \right)^3, \quad \dots \quad \dots \quad (5) \end{aligned}$$

since  $x = s'$  approximately.

From (4) and (5)

$$\begin{aligned} \frac{d^5}{48c^4} &= \frac{1}{6c_1^3} \left( x + \frac{d}{2} \right)^3 \\ \therefore c_1^3 &= \frac{8c^4}{d^5} \left( x + \frac{d}{2} \right)^3 \\ &= 8c^4 \left( \frac{x}{d} + \frac{1}{2} \right)^3 = c^3 \left( 1 + \frac{2x}{d} \right)^3 \\ \therefore c_1 &= c \left( 1 + \frac{2x}{d} \right)^{\frac{3}{2}} = c \left( 1 + \frac{3x}{d} \right) \text{ approximately} \\ &= c \left( 1 + \frac{3s'}{d} \right). \end{aligned}$$

The sag in the first case  $= \frac{d^3}{8c}$ , [Cf. Art 58 Eq. (5).]

$$\begin{aligned} \text{and in the second, } &= \frac{\left( \frac{d}{2} + s' \right)^3}{2c_1} = \frac{1}{2c} \frac{\left( \frac{d}{2} + s' \right)^3}{1 + \frac{3s'}{d}} \\ &= \frac{d^3}{8c} \left( 1 + \frac{2s'}{d} \right)^3 \left( 1 + \frac{3s'}{d} \right)^{-3} \\ &= \frac{d^3}{8c} \left( 1 + \frac{4s'}{d} \right) \left( 1 - \frac{3s'}{d} \right) \text{ approximately} \end{aligned}$$

$$\begin{aligned}
 &= \frac{d^2}{8c} \left( 1 + \frac{s'}{d} \right), \quad \text{since } \left( \frac{s'}{d} \right)^2 \text{ is negligible} \\
 &= \left( 1 + \frac{s'}{d} \right) \text{ times the original sag.}
 \end{aligned}$$

**59. The Parabolic Catenary.** In the common catenary the load per unit length of chain was constant.

We will now assume a chain so loaded that the weight supported per unit length is proportional to the horizontal projection of that length.

This is similar to the case of a chain of negligible weight supporting a bridge connected to it by vertical rods, the bridge being of uniform weight per unit length.

Take the lowest point A of the chain as origin and the  $x$  axis horizontal.

Consider the equilibrium of a finite portion AP of the chain.

Let  $T_0$  be the tension at A,  $T$  the tension at P and  $\psi$  the inclination at P. Then, if  $w$  is the load per unit of horizontal span, the vertical force on AP is  $w x$ .

$$\text{Hence} \quad T \sin \psi = w x \quad \dots \dots \dots (1)$$

$$\text{Also} \quad T \cos \psi = T_0 \quad \dots \dots \dots (2)$$

$$\text{Dividing,} \quad \tan \psi = \frac{w x}{T_0} \quad \dots \dots \dots (3)$$

$$\text{i.e.} \quad \frac{dy}{dx} = \frac{w x}{T_0} \quad \dots \dots \dots (4)$$

$$\text{Integrating} \quad y = \frac{1}{2} \frac{w}{T_0} x^2 + C \quad \dots \dots \dots (5)$$

But  $y = 0$  when  $x = 0$ , so that  $C$  vanishes

$$\text{and} \quad y = \frac{1}{2} \frac{w}{T_0} x^2. \quad \dots \dots \dots (6)$$

The form of the chain is therefore a parabola.

If we write  $T_0 = w c$ , then

$$x^2 = 2 c y \quad \dots \dots \dots (7)$$

a parabola with latus rectum  $2c$ .

From equations (1) and (2),

$$T^2 = T_0^2 + w^2 x^2$$

$$\text{or, using eqn. (6)} = \frac{w^2 x^4}{4 y^2} + w^2 x^2 = \frac{w^2 x^2}{y^2} \left( \frac{x^2}{4} + y^2 \right).$$

$$\therefore \quad T = \frac{w x}{y} \sqrt{\left( \frac{1}{4} x^2 + y^2 \right)} \quad \dots \dots \dots (8)$$

or, using eqn. (7)

$$T = w\sqrt{(2cy + c^2)}. \quad \dots \quad (9)$$

**Ex. X.** Find, in the case of the parabolic catenary suspended between two points in the same horizontal plane, the tension at the middle point in terms of the total load ( $W$ ) on the chain and of the ratio ( $r$ ) between dip and span.

It is required to determine  $T_0$ .

Let the span be  $a$  and the dip  $h$ .

Then from eqn. (6) above

$$h = \frac{1}{2} \frac{w}{T_0} \frac{a^2}{4} \quad \dots \quad (1)$$

But  $W = aw$  i.e.  $w = \frac{W}{a}$

$$\therefore h = \frac{1}{2} \frac{W}{T_0} \frac{a}{4}$$

$$\text{i.e. } T_0 = \frac{W}{8r} \quad \text{where } r = \frac{h}{a}.$$

**60. Catenary of Uniform Strength.** In the case of a uniform chain we have shown that the tension at any point is equal to  $wy$ , where  $y$  is the distance of the point from the directrix. Thus the tension increases from the lowest point and is greatest at the point of attachment, so that there is a limit to the possible span for a chain of any given material and given cross-section. If the cross-section at every point

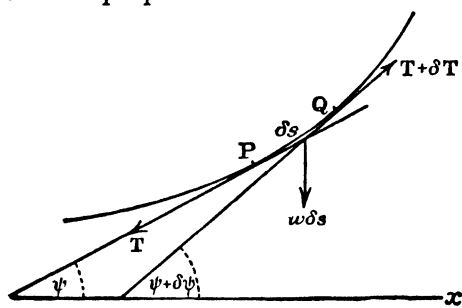


Fig. 110.

a chain called the catenary of uniform strength as the tendency to break is the same for every point of the chain.

Consider the equilibrium of an element PQ of the chain of length  $ds$ , and of weight  $wds$ .

Resolving along, and perpendicular to the tangent at P.

$$(T + \delta T) \cos \delta\psi - T = wds \sin \psi \quad \dots \quad (1)$$

$$\text{and} \quad (T + \delta T) \sin \delta\psi = wds \cos \psi. \quad \dots \quad (2)$$

When Q approaches indefinitely close to P, equations (1) and (2) become

$$\frac{dT}{ds} = w \sin \psi \quad \dots \quad (3)$$

and 
$$T \frac{d\psi}{ds} = w \cos \psi. \quad \dots \quad \dots \quad (4)$$

Now, the conditions of the question require that  $w$  should vary as the tension  $T$ , so that

$$T = w\lambda \quad \dots \quad \dots \quad (5)$$

where  $\lambda$  is constant.

Substituting in equation (4)

$$\frac{ds}{d\psi} = \lambda \sec \psi. \quad \dots \quad \dots \quad (6)$$

Integrating, we have  $s = \lambda \log (\sec \psi + \tan \psi) \quad \dots \quad (7)$   
 where the arbitrary constant is zero, if  $s = 0$  when  $\psi = 0$ .

Now, for any curve,  $\sin \psi = \frac{dy}{ds}$  and  $\cos \psi = \frac{dx}{ds}$   
 so that equation (4) may be written

$$T \frac{d\psi}{ds} = w \frac{dx}{ds}$$

$$\lambda \int d\psi = \int dx \quad \therefore \quad \psi = \frac{x}{\lambda}. \quad \dots \quad \dots \quad (8)$$

Substituting this value of  $\psi$  in equation (7), we have

$$s = \lambda \log \left( \sec \frac{x}{\lambda} + \tan \frac{x}{\lambda} \right). \quad \dots \quad \dots \quad (9)$$

which may also be written

$$s = \lambda \log \tan \left( \frac{\pi}{4} + \frac{x}{2\lambda} \right). \quad \dots \quad (10)$$

Again, 
$$\frac{dy}{dx} = \tan \psi = \tan \frac{x}{\lambda}$$

$$\therefore \quad y = \lambda \log \sec \frac{x}{\lambda} \quad \dots \quad \dots \quad \dots \quad (11)$$

where again the arbitrary constant to be added is zero, if the origin be taken at the lowest point of the curve.

Investigating the form of the curve given by equation (11) we see that the curve is symmetrical about the 'y' axis and that  $y$  increases as  $x$  increases.

If  $\frac{x}{\lambda} = \pm \frac{\pi}{2}$ , then  $\sec \frac{x}{\lambda} = \infty$  and  $y = \infty$ , so that there are two vertical asymptotes given by  $x = \pm \frac{\lambda\pi}{2}$  and the maximum span must therefore be less than  $\lambda\pi$ .

To get an expression for the tension at any point, we may make use of equation (3).

Since  $\sin \psi = \frac{dy}{ds}$

$$\therefore \frac{dT}{dy} = w = \frac{T}{\lambda},$$

whence  $\int \frac{dT}{T} = \int \frac{dy}{\lambda}$

or  $\log T = \frac{y}{\lambda} + C.$

If the tension is  $T_0$  at the vertex,  $C = \log T_0$

and  $\therefore \log \frac{T}{T_0} = \frac{y}{\lambda}.$

$$\therefore T = T_0 e^{\frac{y}{\lambda}}.$$

But from equation (II)  $e^{\frac{y}{\lambda}} = \sec \frac{x}{\lambda}.$

$$\therefore T = T_0 \sec \frac{x}{\lambda} = T_0 \sec \psi = T_0 \frac{\rho}{\lambda}$$

from equation (6), so that the tension at any point is proportional to the radius of curvature at that point.

Again from equation (9) we have

$$e^{\frac{s}{\lambda}} = \sec \frac{x}{\lambda} + \tan \frac{x}{\lambda},$$

$$\therefore e^{-\frac{s}{\lambda}} = \sec \frac{x}{\lambda} - \tan \frac{x}{\lambda},$$

$$\therefore \frac{e^{\frac{s}{\lambda}} + e^{-\frac{s}{\lambda}}}{2} = \sec \frac{x}{\lambda}, \quad (12)$$

$$\therefore T = T_0 \cosh \frac{s}{\lambda}. \quad \dots \dots \dots (13)$$

### Problems VIII.

1. A sample of wire weighs 1 lb. per yd. It is tested and found to break under a tension of 600 lb. A piece of this wire is bent into a circle and spun in its own plane about its centre. Show that it will break when its speed is 240 ft. per sec. whatever the size of the circle.

2. A heavy uniform chain rests in limiting equilibrium on the rough cycloid  $s = 4a \sin \phi$  in a vertical plane, vertex upwards.

One end of the chain is at the vertex and the other at the cusp.

Prove that the coefficient of friction,  $\mu$ , is given by  $3e^{-\frac{\mu\pi}{2}} = 1 + \mu^2$ .

3. Two weights  $W$  and  $w$  are connected by a light inelastic string which passes over a rough ( $\mu$ ) cylindrical beam whose axis is horizontal. If  $W > w.e^{\pi\mu}$ , find the acceleration of the masses when free to move.

4. Two rough cylindrical beams have their axes horizontal on the same horizontal level and parallel, at a distance  $d$  apart. The radii are  $a$ ,  $a_1$  and the co-efficients of friction  $\mu$ ,  $\mu_1$ . Weights  $W$  and  $W_1$  are connected by a light inelastic string which passes over both beams. Find the conditions under which there would be equilibrium.

5. A length of uniform wire is bent into a circle and is then caused to rotate in its own plane about the centre of the circle. Show that the wire will break when its linear velocity is equal to  $\sqrt{gh}$ , where  $h$  is the maximum length of the wire which can be hung vertically from one end.

6. Determine the motion of a circular hoop of radius  $a$  feet, whirling in a vertical plane on a round horizontal stick, if released when the centre is moving with velocity  $V$  feet per second at an angle  $\alpha$  with the horizon; and prove that it will make  $V/2\pi a$  revolutions per second in the air.

Prove that the tension in the hoop will be the weight of a length  $V^2/g$  feet of the rim (the tension length).

7. Masses of 50 lb. and 80 lb. are connected by a light string which passes over a rough cylindrical beam. If the system be in limiting equilibrium, find the co-efficient of friction. Find also what mass must be added to the 50 lb. in order that the mass of 80 lb. should be on the point of moving upwards.

8. A rough cylindrical beam has a light string wound round it, the co-efficient of friction being  $\cdot 15$ . How many times must the string be wound round in order that a force of 100 lb. at one end will cause the string to be on the point of slipping when a force of 1 lb. wt. is applied at the other end?

9. A thin cylindrical boiler, radius 4 feet, rests with its base horizontal and contains steam at pressure 120 lb. wt. per square inch. Find the tension on a horizontal strip of the curved surface, 1 inch wide.

10. A belt driven pulley of diameter 2 feet transmits 10 horsepower when running at 240 revolutions per minute. If the belt is just on the point of slipping, and subtends an arc of  $180^\circ$  of the pulley, find the biggest tension of the belt, the co-efficient of friction being  $0\cdot 4$ .

11. A thin circular hoop, radius 2 feet and mass 12 oz. rotates 200 times per minute about its centre, in a horizontal plane. What is the tension? If the tensile strength of the wire be 12 lb. wt., what is the greatest rate at which it could rotate?



12. Find the maximum tension of a wire, which weighs  $1\frac{1}{2}$  lb. per foot, and hangs with a sag of six inches in a span of 60 feet.

13. A telegraph wire has a span of 66 ft. ; find the sag in the middle if the tension is not to exceed 150 lb., assuming that 20 ft. of the wire weigh 1 lb.

14. A uniform wire 75 ft. long has a horizontal span of 50 ft. ; find (graphically) the droop in the middle.

15. A telegraph wire is stretched between two posts 50 yards apart and the sag in the middle is one yard. What is the length of the wire and the tension if one yard of wire weighs 3 ounces ?

16. The weight per foot run of a telegraph wire is  $w$  lb., the length of wire between the posts is  $l$  feet, a weight  $W$  lb. is hung at the middle, and the wire sags  $x$  feet in the middle. Prove that the tensions at the posts is

$$w \left\{ \frac{l^2}{8x} \left( 1 + \frac{2W}{lw} \right) + \frac{1}{2} x \right\} \text{ lb.}$$

17. A uniform chain hangs between two fixed points, and various weights are attached to it at intermediate points. Prove that the intervening portions of the chain form arcs of equal catenaries.

18. In the common catenary show that the weight of the string between the lowest point and any other point is the geometrical mean between the sum and difference of the tensions at the two points.

19. A chain weighing 5 lb. per foot and 200 ft. long is hung between one point and another at a level 20 ft. below the first ; the lowest point of the chain is 10 ft. below the lower point of attachment. Find the maximum tension in the chain.

20. A uniform chain is stretched from a point on the ground to a point 10 ft. above the ground. The tension at the ground is that of the weight of 3100 ft. of chain, and the inclination there of the tangent to the horizon is  $\cos^{-1} \left( \frac{16}{31} \right)$ . Find the length of the chain to the nearest inch.

21. Show that the length of an endless chain which will hang over a circular pulley of radius  $a$  so as to be in contact with  $2/3$  of the circumference of the pulley, is

$$a \left[ \frac{3}{\log_e (2 + \sqrt{3})} + \frac{4\pi}{3} \right].$$

22. A heavy uniform chain hangs in the form of a catenary from two pegs  $P$  and  $Q$  at the same level.  $P'$  and  $Q'$  are the two points of the catenary where the tangents make with the vertical acute angles which are twice the acute angles made with the vertical by the tangents at  $P$  and  $Q$ . Show that the chain could hang over two smooth pegs at  $P'$  and  $Q'$  with its extremities hanging freely and vertically and the portion between the pegs hanging in the same catenary as before.

Show also that the resultant pressures on the pegs at  $P'$  and  $Q'$  would be the same as those on the pegs at  $P$  and  $Q$ .

23. A man holds in his hands the ends A and B of a uniform chain of weight  $W$  and length  $2l$ . Initially his hands are close together and he pulls them apart in the same horizontal line till the tangents at the ends of the chain are inclined at an angle  $\alpha$  to the horizontal; show that the work he does is

$$\frac{Wl}{2} \left[ 1 - \operatorname{cosec} \alpha + \cot^2 \alpha \log (\tan \alpha + \sec \alpha) \right].$$

24. A uniform chain of length  $l$  rests in a straight line on a rough horizontal table. One end is raised to a height  $h$  above the table and the chain is on the point of motion. Show that the length of the straight part on the table is

$$l + \mu h - \{(\mu^2 + 1)h^2 + 2\mu h\}^{\frac{1}{2}}.$$

25. An elastic string (weight  $w$  per unit length) natural length  $2a$ , is stretched from two points in the same horizontal level distant  $2a$  apart. If the modulus ( $\lambda$ ) be very large, prove that the

tension is  $w \sqrt{\frac{a^3}{6 \sqrt{\frac{w^2 a^6}{6 \lambda^2}}}}$  and the sag  $\frac{1}{2} \sqrt{6a^3 \sqrt{\frac{w^2 a^6}{6 \lambda^2}}}.$

26. The sag at the middle point of a uniform string, suspended from two points in the same horizontal line is  $\frac{1}{n}$ th of the length of the string. Prove that the ratio of the tension at either end to the weight of the string is equal to  $(n^2 + 4)/8n$ .

27. A chain hangs in equilibrium over two smooth points which are in a horizontal straight line and at a given distance apart; find the least length of the chain that equilibrium may be possible.

28. A heavy chain rests in equilibrium in a vertical plane supported by two smooth pegs A and B not in the same horizontal line; prove that the free ends of the chain lie on the directrix of the catenary formed by the part of the chain between A and B.

If  $x$  and  $y$  are the lengths of chain hanging vertically from A and B, and  $z$  is the length of chain between A and B, and if  $\alpha$  and  $\beta$  are the angles the tangents to the chain at A and B make with the vertical, prove that

$$\frac{x}{\sin \beta} = \frac{y}{\sin \alpha} = \frac{z}{\sin (\alpha + \beta)}.$$

29. Prove that if a chain ABC fastened at A is led over a pulley B, so as to rest on a smooth inclined plane BC, the part AB will assume a catenary curve in which the tension at any point P will be the same as at Q at the same level on BC; and deduce the analytical properties of the catenary.

Prove that if the plane is rough the length of the chain BC may be altered without affecting AB, within the limits

$$BC \sin \alpha \cos \phi \operatorname{cosec} (\alpha \pm \phi)$$

$\alpha$  denoting the slope of BC and  $\phi$  the limiting angle of friction.

30. If  $l$  is the length of a uniform string and  $a, b$  the horizontal and vertical distances between its ends, show that the parameter  $c$  of the curve in which the string will hang when supported by its ends is given by the equation

$$l^2 - b^2 = 4c^2 \sinh^2 \frac{a}{2c},$$

and that the vertex is at a horizontal distance.

$$\frac{1}{2}c \log \{(l+b)/(l-b)\}$$

from the mid-point of the line joining the ends.

31. Find the tensions at the points of support of a suspension bridge whose weight is 150 tons evenly distributed over the span of 120 feet the depth of the lowest point of the chain below the points of suspension being 15 feet.

32. A uniform chain of length  $2l$  and weight  $w$  per unit of length is suspended from two fixed points at the same level at a distance  $2a$  apart, and a weight  $W$  is attached at its middle point. Prove that the parameter  $c$  of the catenary in which either portion of the chain hangs is given by the equation

$$\frac{W}{2w} \left( \cosh \frac{a}{c} - 1 \right) + \sqrt{\left( c^2 + \frac{W^2}{4w^2} \right)} \cdot \sinh \frac{a}{c} = l$$

and that, if  $W$  is so great that both portions of the chain are nearly straight, the value of  $c$  is approximately

$$\frac{1}{2} \left( \frac{W}{w} \cot \alpha + a \operatorname{cosec} \alpha \right),$$

where  $a = l \cos \alpha$ ,  $0 < \alpha < \frac{\pi}{2}$ , and terms of the order  $\frac{wa^2}{W}$  are neglected.

33. Deduce the equation of the parabolic catenary from the fact that, since the tensions at the lowest point  $C$  and any other point  $P$  support the weight of the bridge vertically below these two points, the tangents at  $C$  and  $P$  must intersect vertically above the centre of gravity of the portion of the bridge supported.

34. Prove that in the catenary of uniform strength the depth of the centre of gravity of any arc below the intersection of the normals at its extremities is constant and equal to  $\lambda$ .

35. The length of the chain in the case of the catenary of uniform strength is  $l$ . If the chain hang from two supports in the same horizontal line at a distance  $a$  apart, prove that

$$\tanh \frac{l}{4\lambda} = \tanh \frac{a}{4\lambda}.$$

## CHAPTER IX

### MOTION OF RIGID BODIES UNDER THE ACTION OF FORCES IN ONE PLANE

**61. D'Alembert's Principle.** This principle follows as a result of Newton's Third Law of Motion and states that the internal actions and reactions of any system of rigid bodies in motion are in equilibrium among themselves.

Any one particle of a moving rigid body is acted upon by **impressed** (i.e. external) forces and **molecular** forces due to the other particles. The single force on the particle which would cause its motion to be precisely the same is called the **effective** force on the particle. If  $F$  be the resultant impressed force on the particle,  $R$  the resultant molecular force,  $m$  the mass of the particle and  $a$  its acceleration, then the forces  $F$ ,  $R$  and  $-ma$  (i.e., the effective force reversed) will be in equilibrium. Considering all the particles of the rigid body, the forces  $\Sigma F$ ,  $\Sigma R$ , and  $-\Sigma ma$  will be in equilibrium, but by D'Alembert's Principle,  $\Sigma R = 0$  and thus if forces equal to the reversed effective forces be applied to each particle of the system, these would be in equilibrium with the impressed forces.

**62. Equations of Motion.** It is proved in elementary statics that a system of forces in one plane acting on a rigid body can be reduced to a single force acting through any given point together with a couple whose axis is perpendicular to the plane and passes through the given point. The molecular reactions of course will be differently distributed but in each case their algebraic sum will be zero and thus the external effect will be the same.

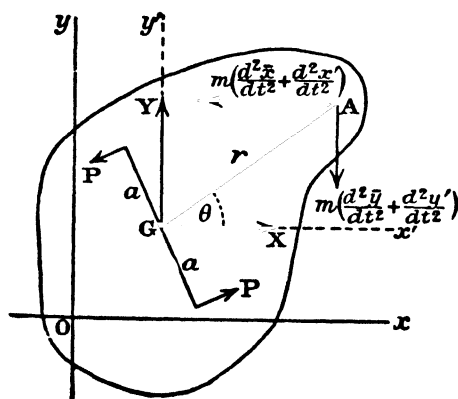


Fig. III.

Let  $G$  be the centre of inertia of a rigid body and let  $\bar{x}$ ,  $\bar{y}$  be the co-ordinates of  $G$  referred to fixed axes,  $Ox$  and  $Oy$  at right angles. Suppose the system of forces acting on the rigid body at any instant to be reduced to a single force acting at  $G$  whose components parallel to the axes are  $X$  and  $Y$ , together

with a couple whose moment (or Torque) is  $2Pa$  and whose axis passes through G. In general  $X$ ,  $Y$  and  $2Pa$  vary with the time  $t$  but there is no loss of generality by supposing the arm  $2a$  of the couple to be constant as  $P$  varies.

A particle of mass  $m$  at A whose co-ordinates referred to parallel axes through G are  $x'$ ,  $y'$  will have accelerations

$$\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2} \text{ and } \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}$$

and thus the forces  $X$ ,  $Y$ ,  $-\Sigma m\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}\right)$ ,  
 $-\Sigma m\left(\frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}\right)$

together with the torque  $2Pa$  will form a system in equilibrium. Resolving along the axes and taking moments about G, we obtain the equations

$$X = \Sigma m\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}\right) \quad \dots \quad \dots \quad (1)$$

$$Y = \Sigma m\left(\frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}\right) \quad \dots \quad \dots \quad (2)$$

$$\text{and } 2Pa = \Sigma m\left[x'\left(\frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}\right) - y'\left(\frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}\right)\right] \dots \quad (3)$$

Now as G is the centre of inertia,  $\Sigma mx' = 0$

$$\therefore \Sigma m \frac{d^2x'}{dt^2} = 0 \quad \text{and} \quad \text{similarly} \quad \Sigma m \frac{d^2y'}{dt^2} = 0.$$

Also  $\frac{d^2\bar{x}}{dt^2}$  is the same for every particle of the body and thus,

if  $M$  be the whole mass of the body, equation (1) becomes

$$X = M \frac{d^2\bar{x}}{dt^2} \quad \dots \quad \dots \quad (4)$$

$$\text{Similarly} \quad Y = M \frac{d^2\bar{y}}{dt^2} \quad \dots \quad \dots \quad (5)$$

Also, equation (3) becomes

$$2Pa = \frac{d^2\bar{y}}{dt^2} \cdot \Sigma mx' - \frac{d^2\bar{x}}{dt^2} \cdot \Sigma my' + \Sigma m\left[x' \frac{d^2y'}{dt^2} - y' \frac{d^2x'}{dt^2}\right]$$

$$\text{i.e.} \quad 2Pa = \Sigma m\left[x' \frac{d^2y'}{dt^2} - y' \frac{d^2x'}{dt^2}\right] \dots \quad (6)$$

Equations (4) and (5) show that the centre of inertia of a body moves as if all the mass were collected there and that its acceleration is independent of the angular acceleration of the body.

Equation (6) is the same equation as would be obtained if  $G$  were fixed; hence the motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same external forces acted on the body.

These two results show that the motions of translation and rotation can be considered independently.

Equation (1) can be written  $\frac{d}{dt} \left[ \Sigma m \frac{dx}{dt} \right] = X$  where  $x = \bar{x} + x'$  and thus the rate of change of linear momentum in any direction equals the sum of the impressed forces in that direction.

Also by taking moments about  $O$ , we have

$$\begin{aligned} Y\bar{x} - X\bar{y} + 2Pa &= \Sigma m \left( x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) \\ &= \frac{d}{dt} \left[ \Sigma m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right]. \end{aligned}$$

Thus the rate of change of the moment of momentum about any point equals the sum of the moments of the impressed forces about that point.

Let the polar co-ordinates of  $A$  referred to the axes through  $G$  be  $r, \theta$ ; then,  $G$  being fixed, the accelerations of the particle at  $A$  are  $r \frac{d^2\theta}{dt^2}$  perpendicular to  $GA$  and  $r \left( \frac{d\theta}{dt} \right)^2$  along  $AG$ , for, the body being rigid,  $r$  is constant for the particle under consideration. Thus the moment of the effective force acting at  $A$  is  $mr^2 \frac{d^2\theta}{dt^2}$

$$\therefore 2Pa = \Sigma mr^2 \frac{d^2\theta}{dt^2}.$$

Now  $\frac{d^2\theta}{dt^2}$  is the same for every particle as the body is rigid and  $\Sigma mr^2 = Mk^2$  where  $k$  is the radius of gyration about an axis through  $G$  perpendicular to the plane; hence equation (6) becomes

$$2Pa = Mk^2 \frac{d^2\theta}{dt^2} \dots \dots \dots (7)$$

**63. Work and Energy.** Referring to the figure of the previous article, let  $x, y$  be the co-ordinates of  $G$  referred to fixed axes, at time  $t$ . At time  $t_1$  suppose the co-ordinates of  $G$  to be  $x_1, y_1$ ; its velocity parallel to the axes,  $u_1, v_1$ ; its angular velocity,  $\omega_1$ ; and  $\theta_1 = \widehat{AGx'}$ . Also suppose that

at time  $t_2$ , the corresponding quantities are  $x_2, y_2; u_2, v_2; \omega_2$ ; and  $\theta_2$ .

The total work done by the impressed forces from  $t = t_1$ , to  $t = t_2$  is

$$\int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy + 2a \int_{\theta_1}^{\theta_2} P d\theta.$$

Now  $X = M \frac{d^2x}{dt^2} = Mu \cdot \frac{du}{dx}$  where  $u = \frac{dx}{dt}$ .

Also  $Y = Mv \frac{dv}{dx}$  where  $v = \frac{dy}{dt}$

and  $2aP = Mk^2 \frac{d^2\theta}{dt^2} = Mk^2 \omega \cdot \frac{d\omega}{d\theta}$  where  $\omega = \frac{d\theta}{dt}$ .

$$\begin{aligned} \therefore \int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy + 2a \int_{\theta_1}^{\theta_2} P d\theta &= M \int_{u_1}^{u_2} u du + M \int_{v_1}^{v_2} v dv + Mk^2 \int_{\omega_1}^{\omega_2} \omega d\omega \\ &= \frac{1}{2} M (u_2^2 - u_1^2) + \frac{1}{2} M (v_2^2 - v_1^2) + \frac{1}{2} Mk^2 (\omega_2^2 - \omega_1^2) \\ &= \text{total gain of kinetic energy of the body.} \end{aligned}$$

**Ex. I.** One end of a light string is fixed to a point on the rim of a uniform circular disc, radius  $a$  and mass  $M$ , and the string is wound round the rim several times. The free end of the string is attached to a fixed point and the disc is held so that the part of the string not in contact with it is vertical. If the disc be let go, find the acceleration and the tension of the string.

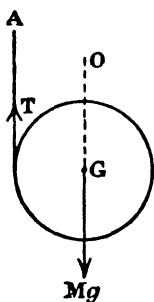


Fig. 112.

Let A be the fixed point and O the initial position of the centre G of the disc. Let T be the tension of the string. As there is no force with a horizontal component, G will move vertically downwards.

Let  $x$  be the distance moved by G and  $\theta$  the angle turned through by a certain radius in time  $t$ , then

$$Mg - T = M \frac{d^2x}{dt^2} \quad \dots \quad (1)$$

and  $T \cdot a = M \cdot \frac{a^2}{2} \cdot \frac{d^2\theta}{dt^2} \quad \dots \quad (2)$

Also from the geometrical conditions of the question,  $x = a\theta$ , hence

$$\frac{d^2x}{dt^2} = a \frac{d^2\theta}{dt^2} \quad \dots \quad (3)$$

Eliminating T and  $\frac{d^2\theta}{dt^2}$  from these equations, we have

$$Mga = Ma \frac{d^2x}{dt^2} + M \cdot \frac{a}{2} \frac{d^2x}{dt^2},$$

$$\therefore \frac{d^2x}{dt^2} = \frac{2}{3}g.$$

Substituting in equation (1),  $T = \frac{1}{3}Mg$ .

**Ex. II.** Two unequal masses,  $m_1$  and  $m_2$  ( $m_1 > m_2$ ) are suspended by a light string over a pulley of mass  $M$  and radius  $a$ .

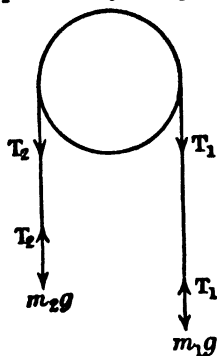


Fig. 113.

There is no slipping and the friction of the axle may be neglected. Show that the acceleration of the masses is constant: if the acceleration be  $f$ , find the radius of gyration of the pulley about the axle.

Let  $T_1, T_2$  be the tensions of the portions of the strings and let  $x$  be the distance moved by each mass in time  $t$  and  $\theta$  the angle through which the pulley rotates in the same time; then

$$m_1 g - T_1 = m_1 \frac{d^2 x}{dt^2} \quad \dots \quad (1)$$

$$T_2 - m_2 g = m_2 \frac{d^2 x}{dt^2} \quad \dots \quad (2)$$

Also if  $k$  be the radius of gyration of the pulley about the axle,

$$(T_1 - T_2)a = M k^2 \frac{d^2 \theta}{dt^2} \quad \dots \quad (3)$$

Also as  $x = a\theta$ , then  $\frac{d^2 x}{dt^2} = a \frac{d^2 \theta}{dt^2}$  ... .. (4)

Eliminating  $T_1, T_2$  and  $\frac{d^2 \theta}{dt^2}$  from these equations, we have

$$a g (m_1 - m_2) - M \frac{k^2}{a} \frac{d^2 x}{dt^2} = a (m_1 + m_2) \frac{d^2 x}{dt^2}$$

$$\therefore \frac{d^2 x}{dt^2} = \frac{a^2 (m_1 - m_2)}{a^2 (m_1 + m_2) + M k^2} \cdot g, \text{ which is constant.}$$

If  $\frac{a^2 (m_1 - m_2)}{a^2 (m_1 + m_2) + M k^2} \cdot g = f$

then  $k^2 = \frac{a^2}{M f} \left[ (g - f)m_1 - (g + f)m_2 \right].$

**Ex. III.** On a table there is a billiard ball of mass  $m$  and radius  $a$  which initially rotates with angular velocity  $\omega$  about a horizontal diameter and its centre is at rest. Find the distance moved by the ball during the time slipping takes place and find the velocity of the ball when it rolls without slipping (co-efficient of friction of the table =  $\mu$ ).

Let A be the point of contact initially and B, where  $AB = x$ , the point of contact after time  $t$ . Let  $\theta$  be the angle through which the radius OA, rotates in this time.

As friction acts in a direction opposed to that of relative motion, the ball will begin to move along a straight line at right angles to the axis of rotation. There is no motion of the centre G in a vertical direction so that the vertical reaction at the point of contact is  $mg$ .



While slipping takes place, the maximum force of friction,  $\mu mg$  is acting,

hence 
$$\mu mg = m \frac{d^2x}{dt^2} \quad \dots \quad \dots \quad \dots \quad (1)$$

and 
$$\mu mga = m \cdot \frac{2}{5} a^2 \frac{d^2\theta}{dt^2} \cdot \dots \quad \dots \quad \dots \quad (2)$$

From (1), the linear acceleration is constant and equal to  $\mu g$

and the angular acceleration is constant and equal to  $\frac{5\mu g}{2a}$ .

Integrating equations (1) and (2) with respect to  $t$ , and substituting initial conditions  $\frac{dx}{dt} = 0$ ,  $x = 0$ ,  $\frac{d\theta}{dt} = -\omega$ ,  $\theta = 0$ , when  $t = 0$ ,

we have 
$$\frac{dx}{dt} = \mu gt \dots \quad (3) \quad \text{and} \quad x = \frac{1}{2} \mu gt^2 \dots \quad (4)$$

Also 
$$\frac{d\theta}{dt} = \frac{5\mu gt}{2a} - \omega \dots (5) \quad \text{and} \quad \theta = \frac{5\mu gt^2}{4a} - \omega t. \quad (6)$$

Now, at the moment that slipping just ceases to take place,

$$\frac{dx}{dt} = -a \frac{d\theta}{dt}$$

$\therefore$  from equations (3) and (5) the time during which slipping takes place is given by

$$\mu gt = a \left( \omega - \frac{5\mu gt}{2a} \right)$$

i.e. 
$$t = \frac{2a\omega}{7\mu g}.$$

$\therefore$  from equation (4), the distance moved by the ball during which slipping takes place is  $\frac{2a^2\omega^2}{49\mu g}$  and from equation (3) the velocity at the instant slipping ceases is  $\frac{2}{7}a\omega$ .  $\dots \dots (3)$

The motion after slipping ceases is determined by the equations

$$F = m \frac{d^2x}{dt^2} \dots (7) \quad Fa = m \cdot \frac{2a^2}{5} \frac{d^2\theta}{dt^2} \dots (8)$$

and  $\frac{d^2x}{dt^2} = a \frac{d^2\theta}{dt^2} \dots (9)$  where  $F$  is the force of friction. If  $F$

is not zero, these equations give  $a^2 = \frac{2}{5}a^2$  which is impossible. Therefore friction ceases to act as soon as slipping ceases and afterwards the linear velocity and the angular velocity continue to be  $\frac{2}{7}a\omega$  and  $\frac{2}{7}\omega$  respectively.

NOTE I. The number of revolutions made by the ball during the time slipping takes place is from equation (6) equal to

$$\frac{1}{2\pi} \left[ \frac{5a\omega^2}{49\mu g} - \frac{2a\omega^2}{7\mu g} \right] \text{ i.e. } - \frac{9a\omega^2}{98\pi\mu g}.$$

**Ex. IV.** A sphere resting on the top of another sphere which is fixed is slightly displaced. Show that if there be no slipping the upper sphere will leave the lower when the angle the common normal makes with the vertical is  $\cos^{-1} \frac{10}{17}$ .

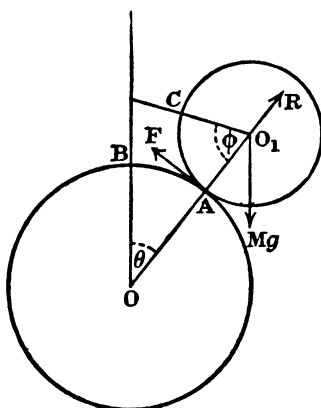


Fig. 114.

OB is the vertical radius ( $b$ ) of the lower sphere and  $O_1C$  the radius ( $a$ ) of the upper which is initially vertical. At the end of time  $t$ , let  $\widehat{BOO_1} = \theta$  and  $\widehat{CO_1A} = \phi$ , then as there is no slipping,  $b\theta = a\phi$  and the upper sphere has rotated through an angle  $\theta + \phi$ , i.e. angle  $\frac{a+b}{a}\theta$ .

The accelerations of  $O_1$  at right angles to  $OO_1$  and along  $O_1O$  are  $(a+b)\frac{d^2\theta}{dt^2}$  and  $(a+b)\left(\frac{d\theta}{dt}\right)^2$  and the angular acceleration of the upper sphere is  $\frac{a+b}{a} \cdot \frac{d^2\theta}{dt^2}$ . Hence if  $R$  be

the normal reaction,  $F$  the force of friction and  $M$  the mass of the upper sphere we have,

$$Mg \sin \theta - F = M(a+b)\frac{d^2\theta}{dt^2} \quad \dots \quad \dots \quad (1)$$

$$Mg \cos \theta - R = M(a+b)\left(\frac{d\theta}{dt}\right)^2 \quad \dots \quad \dots \quad (2)$$

and  $F \cdot a = M \cdot \frac{2a^2}{5} \cdot \frac{a+b}{a} \cdot \frac{d^2\theta}{dt^2} \quad \dots \quad \dots \quad (3)$

Eliminating  $F$  from (1) and (3) we have

$$\frac{d^2\theta}{dt^2} = \frac{5g}{7(a+b)} \sin \theta. \quad \dots \quad \dots \quad (4)$$

Integrating (4) and using the condition that  $\frac{d\theta}{dt} = 0$  when  $t = 0$ ,

we have  $\left(\frac{d\theta}{dt}\right)^2 = \frac{10g}{7(a+b)} (1 - \cos \theta). \quad \dots \quad \dots \quad (5)$

Substituting the values of  $\frac{d^2\theta}{dt^2}$  and  $\left(\frac{d\theta}{dt}\right)^2$  in (1) and (2),

we have  $F = \frac{2}{7} Mg \sin \theta$  and  $R = \frac{1}{7} Mg(17 \cos \theta - 10)$ .

The upper sphere is on the point of leaving the lower when  $R = 0$ ,  
i.e. when  $\theta = \cos^{-1} \frac{10}{17}$ .

**Ex. V.** A sphere makes small oscillations at the bottom of a cylinder in a plane at right angles to the axis which is horizontal. The cylinder being fixed and the surface sufficiently rough to prevent sliding, express  $g$  in terms of  $a$ , the radius of the sphere,  $b$  the radius of the cylinder, and  $T$ , the periodic time of the oscillations.

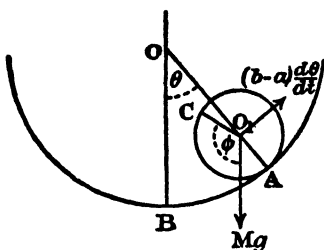


Fig. 115.

lowest position. At the end of time  $t$  from the instant  $O_1C$  is vertical, let  $\widehat{BOO_1} = \theta$  and let  $\phi$  be the angle through which the sphere has rotated, then as there is no slipping

$$a(\phi + \theta) = b\theta.$$

The velocity of  $O_1$  perpendicular to  $OO_1$  is  $(b-a) \frac{d\theta}{dt}$  and the angular velocity of the sphere about its centre is

$$\frac{d\phi}{dt} \text{ i.e. } \frac{b-a}{a} \cdot \frac{d\theta}{dt}.$$

If  $\omega$  be the angular velocity of  $OO_1$  when the sphere is at the lowest position, then

$$Mg(b-a)(1 - \cos \theta) = \frac{1}{2}M \cdot (b-a)^2 \left[ \omega^2 - \left( \frac{d\theta}{dt} \right)^2 \right]$$

$$+ \frac{1}{2}M \cdot \frac{2a^2}{5} \cdot \frac{(b-a)^2}{a^2} \left[ \omega^2 - \left( \frac{d\theta}{dt} \right)^2 \right]$$

$$\text{i.e. } g(1 - \cos \theta) = \frac{7(b-a)}{10} \left[ \omega^2 - \left( \frac{d\theta}{dt} \right)^2 \right].$$

Differentiating with respect to  $t$ , we have

$$g \sin \theta = - \frac{7(b-a)}{5} \cdot \frac{d^2\theta}{dt^2}$$

$$\therefore \frac{d^2\theta}{dt^2} = - \frac{5g}{7(b-a)} \theta \text{ when the oscillations are small.}$$

$$\therefore T = 2\pi \sqrt{\frac{7(b-a)}{5g}} \text{ and thus } g = \frac{28\pi^2(b-a)}{5T^2}.$$

**NOTE. 2.** The above equation of motion is obtained by equating the work done against gravity to the loss of kinetic energy and is used because the forces of reaction and friction (which do no work during the motion) are not required.

The same expression for  $\frac{d^2\theta}{dt^2}$  would have been obtained by introducing these forces and then eliminating them from the three equations of motion. Instead of introducing  $\omega$  the equation could have been written

$$-Mg(b-a) \cos \theta = \text{Constant} - \frac{1}{2}M(b-a)^2 \left( \frac{d\theta}{dt} \right)^2 - \frac{1}{2}M \frac{2a^2}{5} \cdot \frac{(b-a)^2}{a^2} \cdot \left( \frac{d\theta}{dt} \right)^2.$$

**Ex. VI.** A thin uniform rod of length  $2a$  is held horizontally, and is in contact with a rough peg at a point a distance  $c$  from the middle point of the rod. The rod is then released, the peg being fixed: find the angle which the rod makes with the horizontal when slipping commences,  $\mu$  being the coefficient of friction. Also find the initial reaction of the peg.

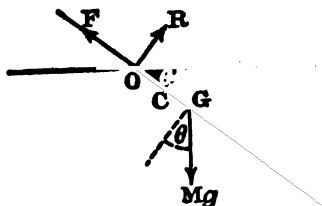


Fig. 116.

Let  $\theta$  be the angle which the rod makes with the horizontal after time  $t$  and let  $R$  be the reaction at the peg and  $F$  the force of friction.

$$\text{Then} \quad Mg \cos \theta - R = Mc \frac{d^2 \theta}{dt^2} \quad \dots \quad (1)$$

$$F - Mg \sin \theta = Mc \left( \frac{d\theta}{dt} \right)^2. \quad \dots \quad (2)$$

Also the equation of work and energy gives

$$Mg c \sin \theta = \frac{1}{2} M \left( c^2 + \frac{a^2}{3} \right) \left( \frac{d\theta}{dt} \right)^2 \quad \dots \quad (3)$$

$$\text{from which} \quad Mg c \cos \theta = M \left( c^2 + \frac{a^2}{3} \right) \frac{d^2 \theta}{dt^2} \quad \dots \quad (4)$$

Subst. the expressions for  $\left( \frac{d\theta}{dt} \right)^2$  and  $\frac{d^2 \theta}{dt^2}$  in (1) and (2), we have

$$R = Mg \cdot \frac{a^2}{3c^2 + a^2} \cos \theta \quad \text{and} \quad F = Mg \cdot \frac{9c^2 + a^2}{3c^2 + a^2} \sin \theta.$$

When slipping is just about to take place,  $\mu = \frac{F}{R}$ .

$$\therefore \text{slipping begins to take place when } \tan \theta = \frac{\mu a^2}{9c^2 + a^2}.$$

Also when  $\theta = 0$ , the reaction is  $\frac{Mga^2}{3c^2 + a^2}$ .

**NOTE 3.** When slipping is taking place,  $OG (= r)$  is no longer constant and the equations of motion become

$$M \left( r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \right) = Mg \cos \theta - R$$

$$M \left[ \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] = Mg \sin \theta - \mu R$$

and

$$R \cdot r = M \cdot \frac{a^2}{3} \frac{d^2 \theta}{dt^2}.$$

The methods employed for the solution of these differential equations are beyond the scope of this book.

**Ex. VII.** A uniform rod of mass  $m$  and length  $2a$  is placed vertically with one end in contact with a smooth horizontal table.

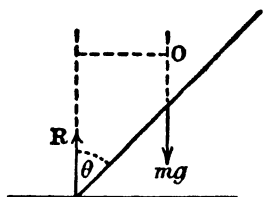


Fig. 117.

It is given a slight displacement and falls: find the reaction when the rod is about to strike the table.

Let  $R$  be the action of the table and  $\theta$  the angle which the rod makes with the vertical after time  $t$ . The only forces acting on the rod are the weight and the reaction which act vertically and consequently the centre of gravity moves in a straight line vertically downwards.

Considering the motion of the centre of gravity, we have

$$mg - R = m \frac{d^2}{dt^2} (a - a \cos \theta)$$

$$\text{i.e.} \quad mg - R = ma \left[ \cos \theta \cdot \left( \frac{d\theta}{dt} \right)^2 + \sin \theta \cdot \frac{d^2\theta}{dt^2} \right] \quad \dots \quad (1)$$

The work and energy equation gives

$$mga(1 - \cos \theta) = \frac{1}{2}m \left( \frac{d\theta}{dt} \right)^2 \left[ \frac{a^2}{3} + a^2 \sin^2 \theta \right]$$

$$\text{i.e.} \quad \left( \frac{d\theta}{dt} \right)^2 = \frac{6g(1 - \cos \theta)}{a(1 + 3 \sin^2 \theta)} \quad \dots \quad \dots \quad (2)$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{3g}{a} \left[ \frac{\sin \theta (7 - 6 \cos \theta - 3 \sin^2 \theta)}{(1 + 3 \sin^2 \theta)^2} \right] \quad (3)$$

By the elimination of  $\left( \frac{d\theta}{dt} \right)^2$  and  $\frac{d^2\theta}{dt^2}$  from (1), (2) and (3) the reaction  $R$  is obtained as a function of  $\theta$ .

When the rod is about to strike the table,

$$\theta = \frac{\pi}{2}; \quad \text{thus} \quad \left( \frac{d\theta}{dt} \right)^2 = \frac{3g}{2a} \quad \text{and} \quad \frac{d^2\theta}{dt^2} = \frac{3g}{4a}$$

and therefore from (1) the reaction then is

$$m \left[ g - \frac{3g}{4} \right] \quad \text{i.e.} \quad \frac{1}{4} mg.$$

**Ex. VIII.** A thin uniform rod of mass  $m$  has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Prove that when the horizontal strain on the hinge is a maximum, the vertical strain is  $\frac{11}{8} mg$ .

Let  $2a$  be the length of the rod and let  $\theta$  be the angle through which it has rotated, after time  $t$ . Then,  $X$  and  $Y$  being the horizontal and vertical strains, we have

$$X = ma \left[ \cos \theta \cdot \left( \frac{d\theta}{dt} \right)^2 + \sin \theta \cdot \frac{d^2\theta}{dt^2} \right] \quad \dots \quad \dots \quad (1)$$

$$Y = mg + ma \left[ \sin \theta \cdot \left( \frac{d\theta}{dt} \right)^2 - \cos \theta \cdot \frac{d^2\theta}{dt^2} \right] \quad \dots \quad (2)$$

The equation of energy and work gives

$$mga \sin \theta = \frac{1}{2} m \left( \frac{d\theta}{dt} \right)^2 \left[ \frac{a^2}{3} + a^2 \right]$$

i.e.  $\left( \frac{d\theta}{dt} \right)^2 = \frac{3g \sin \theta}{2a} \quad \dots \quad \dots \quad (3)$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{3g \cos \theta}{4a} \quad \dots \quad \dots \quad (4)$$

Subst. (3) and (4) in (1) and (2) we have

$$X = \frac{3}{4} mg \sin \theta \cos \theta \quad \dots \quad \dots \quad (5)$$

and  $Y = mg \left[ 1 + \frac{3}{4} (2 \sin^2 \theta - \cos^2 \theta) \right] \quad \dots \quad (6)$

From (5), X is a maximum when  $2\theta = \frac{\pi}{2}$ , i.e. when  $\theta = \frac{\pi}{4}$

$\therefore$  from (6), the vertical strain on the hinge is then  $\frac{11}{8} mg$ .

**64. Motion about a Fixed Axis.** When a rigid body rotates about a fixed axis, the motion of rotation can be determined by two methods without introducing the reaction at the axis. The point of application of this reaction being fixed, no work is done by this force during the motion and hence the equation of work and energy will determine the angular velocity in terms of the other external forces acting on the body. This method has already been applied in some of the worked examples and in Article 52. The second method is to equate the sum of the moments of the external forces about the fixed axis to the sum of the moments of the effective forces about the fixed axis. This follows from the fact that the body under the action of the external forces and the reversed effective forces will be in equilibrium.

If  $m$  be a particle whose distance from the axis is  $r$  and  $\frac{d^2\theta}{dt^2}$  the angular acceleration of the body, the moment of the effective force acting on the particle about the axis will be  $mr^2 \frac{d^2\theta}{dt^2}$ . The sum of the moments of the effective forces is therefore  $\frac{d^2\theta}{dt^2} \cdot \Sigma mr^2$  since, the body being rigid,  $\frac{d^2\theta}{dt^2}$  is the same for every particle. Thus the sum of the moments of the external forces about the axis equals  $Mk_1^2 \frac{d^2\theta}{dt^2}$  where  $Mk_1^2$  is the moment of inertia of the body about the fixed axis.

Referring to figure 95, and applying this equation we have

$$Mgh \sin \theta = - Mk_1^2 \frac{d^2 \theta}{dt^2}$$

$$\text{i.e.} \quad gh \sin \theta = - (k^2 + h^2) \cdot \frac{d^2 \theta}{dt^2},$$

which equation determines the motion of the compound pendulum.

It should be noted that the equation of work and energy is an integration with respect to  $t$  of the equation giving the angular acceleration and thus is especially useful when the latter equation is difficult to integrate.

**Ex. IX.** A uniform thin rod of length  $2a$  and mass  $M$  is rotating on a smooth horizontal plane about one extremity which is fixed. The initial angular velocity is  $\Omega$  and the only retarding force is that of the resistance of the air which is  $kdx$  times the square of the velocity on each element of the rod of length  $dx$ . Find the angular velocity of the rod after time  $t$ .

Let  $\omega$  be the angular velocity of the rod after time  $t$ , then the moment of the air resistance on an element at distance  $x$  from the fixed end, about this end, will be  $kx^3\omega^2 dx$ .

$$\text{Hence} \quad -k\omega^2 \int_0^{2a} x^3 dx = M \left( \frac{a^2}{3} + a^2 \right) \frac{d\omega}{dt}$$

$$\text{i.e.} \quad -4k\omega^2 a^4 = \frac{4}{3} Ma^2 \frac{d\omega}{dt}$$

$$\therefore \quad -3ka^2 dt = M \frac{d\omega}{\omega^2}$$

$$\therefore \quad -3ka^2 t = -\frac{M}{\omega} + \text{Const.}$$

$$\text{Now when} \quad t = 0, \quad \omega = \Omega$$

$$\therefore \quad -3ka^2 t = M \left[ \frac{1}{\Omega} - \frac{1}{\omega} \right]$$

$$\therefore \quad \omega = \frac{M\Omega}{M + 3ka^2\Omega t}$$

**Ex. X.** A uniform thin rod of length  $2a$  is freely hinged at one end and rotates uniformly as a conical pendulum, making  $n$  revolutions per second. Find the angle which the rod makes with the vertical.

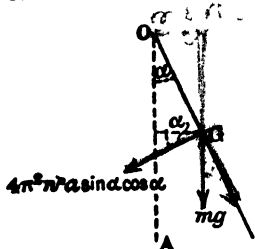


Fig. 118.

OA is the vertical through the fixed end O and G is the middle point of the rod whose mass is  $m$ . Let  $\alpha$  be the angle which the rod makes with the vertical, then at any instant the acceleration of G is  $a \sin \alpha \cdot 4\pi^2 n^2$  horizontally and in the plane containing the rod and OA. The component at right angles to OG is  $4\pi^2 n^2 a \sin \alpha \cos \alpha$  and hence the angular acceleration about O is  $4\pi^2 n^2 \sin \alpha \cos \alpha$  in the vertical plane containing the rod and OA.

Hence 
$$mga \sin \alpha = m \left( \frac{a^2}{3} + a^2 \right) 4\pi^2 n^2 \sin \alpha \cos \alpha.$$

$\therefore \sin \alpha \cdot \left[ g - \frac{16}{3} \cdot a\pi^2 n^2 \cos \alpha \right] = 0$

$\therefore \alpha = 0 \quad \text{or} \quad \cos^{-1} \frac{3g}{16a\pi^2 n^2}.$

If  $3g > 16a\pi^2 n^2$ , i.e. if  $n^2 < \frac{3g}{16a\pi^2}$ , the only possible solution is  $\alpha = 0$ , i.e. the rod hangs vertically. If  $n^2 > \frac{3g}{16a\pi^2}$  then

$$\alpha = \cos^{-1} \frac{3g}{16a\pi^2 n^2}.$$

**65. Impulsive Forces.** Equations (4), (5) and (7) of Article 62 are

$$X = M \frac{d^2 \bar{x}}{dt^2}, \quad Y = M \frac{d^2 \bar{y}}{dt^2} \quad \text{and} \quad 2Pa = Mk^2 \frac{d^2 \theta}{dt^2}.$$

Suppose the forces  $X$ ,  $Y$  and  $P$  are large and act for a small time  $\tau$ . Then the impulses of  $X$  and  $Y$  are  $\int_0^\tau X dt$  and  $\int_0^\tau Y dt$  and the moment of the impulse of the couple  $2Pa$  is  $2a \int_0^\tau P dt$ .

Let  $X_1$ ,  $Y_1$  and  $P_1$  be the impulses of  $X$ ,  $Y$  and  $P$  respectively, then by integration of the above equations with respect to the time  $t$ , from  $t = 0$  to  $t = \tau$ , we have

$$X_1 = M \left[ \frac{d\bar{x}}{dt} \right]_0^\tau, \quad Y_1 = M \left[ \frac{d\bar{y}}{dt} \right]_0^\tau, \quad 2Pa_1 = Mk^2 \left[ \frac{d\theta}{dt} \right]_0^\tau.$$

Suppose that just before the impulses act, the velocities of the centre of inertia of the rigid body are  $u_1, v_1$  parallel to the axes and that  $\omega_1$  is the angular velocity; also that just after the impulses act these velocities become  $u_2, v_2, \omega_2$  respectively. Then the above equations become

$$X_1 = M[u_2 - u_1] \quad \dots \quad \dots \quad (1)$$

$$Y_1 = M[v_2 - v_1] \quad \dots \quad \dots \quad (2)$$

and  $2P_1 a = Mk^2[\omega_2 - \omega_1] \quad \dots \quad \dots \quad (3)$

From equations (1) and (2) we have that the change in the momentum of the mass  $M$  supposed collected at the centre of



gravity in any direction equals the sum of the impulses in that direction and from equation (3), the change in the moment of momentum of the body about its centre of inertia equals the moment about the centre of inertia of the impulses of the forces.

**Ex. XI.** A uniform rod of length  $2l$  lies on a smooth horizontal plane and is freely hinged at a point A at a distance  $a$  from the centre G of the rod. It is struck a blow whose impulse is  $P$  in a direction at right angles to its length at a point B a distance  $b$  from G and  $a + b$  from A. Find the angular velocity imparted and the impulse at the hinge. Find the position of A in order that the impulse at the hinge is zero.

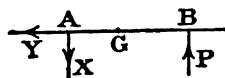


Fig. 119.

Let  $\omega$  be the angular velocity imparted and  $m$  the mass of the rod. Let  $X$  and  $Y$  be the components of the impulse at the hinge in directions perpendicular and parallel to the rod respectively.

$$\text{Then} \quad P - X = m a \omega \quad \dots \quad \dots \quad \dots \quad (1)$$

$$Y = 0 \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\text{and} \quad P b + X a = m \frac{l^2}{3} \omega \quad \dots \quad \dots \quad \dots \quad (3)$$

$$\text{From (1) and (3) we have, } P(a + b) = m \left( a^2 + \frac{l^2}{3} \right) \omega.$$

$$\therefore \quad \omega = \frac{3P(a + b)}{m(3a^2 + l^2)} \quad \text{and} \quad X = P \cdot \frac{l^2 - 3ab}{3a^2 + l^2}.$$

$$\text{In order that } X \text{ should be zero, } a = \frac{l^2}{3b}.$$

**NOTE 4.** The fixed point at which the reaction is zero is known as the Centre of Percussion and thus if the blow were made at one end, the centre of percussion would be at a distance  $\frac{2}{3}l$  from the other end. If the rod were not hinged and if the point A were at rest immediately after the blow equations (1) and (3) would be  $P = m a \omega$  and  $P b = m \frac{l^2}{3} \omega$  from which,  $a = \frac{l^2}{3b}$

and thus the rod would begin to turn about the point distant  $\frac{l^2}{3b}$  from the centre. Subsequently however, there being no forces acting horizontally, G would move with uniform velocity  $\frac{P}{m}$  at right angles to the initial direction of the rod and the rod at the same time would rotate with uniform angular velocity  $\frac{3Pb}{ml^2}$ .

**Ex. XII.** A uniform cube whose mass is  $M$  and whose edges are of length  $a$  rests on a horizontal table and has one edge AB freely hinged to the table. It is struck a blow at the middle point of the edge opposite to AB and in the horizontal direction at right angles to the edge. Find the least impulse of the blow which would cause the cube to topple over.

Let  $P$  be the impulse of the blow and  $\omega$  the angular velocity initially imparted. Then taking moments about  $A$  we have

$$Pa = M \left[ \frac{a^2}{6} + \frac{a^2}{2} \right] \omega.$$

i.e.  $P = \frac{4}{3} a M \omega. \quad \dots \quad \dots \quad \dots \quad (1)$

In order that the centre of gravity of the cube should just reach its highest position we have,

$$\frac{Mga}{2} (\sqrt{2} - 1) = \frac{1}{2} M \left[ \frac{a^2}{2} + \frac{a^2}{6} \right] \omega^2$$

i.e.  $\omega^2 = \frac{3g(\sqrt{2} - 1)}{2a} \quad \dots \quad \dots \quad \dots \quad (2)$

From (1) and (2), the cube will topple over if

$$P > M \sqrt{\frac{2ag(\sqrt{2} - 1)}{3}}.$$

**Ex. XIII.** A uniform lamina mass  $M$  in the shape of an equilateral triangle, side  $2a$ , moves along with velocity  $u$  in a direction parallel to one side. Suddenly the foremost end of this side is fixed. Find the subsequent angular velocity and the impulse of the force at the fixed point.

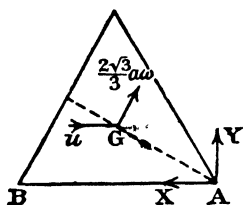


Fig. 120.

Let  $X$ ,  $Y$  be the components of the impulse of the force at the fixed point  $A$  and  $\omega$  the subsequent angular velocity, then

$$M \left( \frac{2\sqrt{3}a\omega}{3} \cos 60^\circ - u \right) = -X \quad (1)$$

$$M \left( \frac{2\sqrt{3}a\omega}{3} \sin 60^\circ \right) = Y \quad \dots \quad \dots \quad (2)$$

$$\text{and } M \frac{a^2}{3} \omega = X \cdot \frac{\sqrt{3}a}{3} - Ya. \quad \dots \quad \dots \quad (3)$$

Eliminating  $X$  and  $Y$ , we have  $\omega = \frac{\sqrt{3}u}{5a}.$

Subst. in (1) and (2), we have

$$X = \frac{4}{5} Mu \quad \text{and} \quad Y = \frac{\sqrt{3}}{5} Mu.$$

$\therefore$  the resultant impulse of the force at  $A$  is  $\frac{\sqrt{19}}{5} Mu$  in a

direction inclined at angle  $\tan^{-1} \frac{\sqrt{3}}{4}$  to the side  $AB$  of the triangle.

**Ex. XIV.** A ball, radius  $a$  and radius of gyration about a diameter  $k$ , rotates with angular velocity  $\omega$  about an axis perpendicular to the plane of motion of its centre and strikes the ground with velocity  $V$  so that just before impact its direction makes an angle  $\alpha$  with

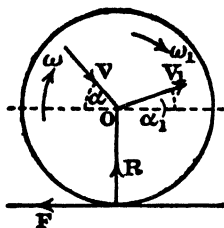


Fig. 121.

the horizontal. Assuming that no slipping takes place find the velocity and the angular velocity immediately after impact,  $e$  being the co-efficient of restitution.

Let  $R$  be the normal impulsive reaction and  $F$  the impulsive friction. Let  $\omega_1$  and  $V_1$  be the angular and linear velocities just after the impact and  $\alpha_1$  the angle that the direction of the velocity of  $O$ , the centre of the ball, makes with the horizontal;

then 
$$M[V_1 \cos \alpha_1 - V \cos \alpha] = -F \quad \dots \quad (1)$$

$$M[V_1 \sin \alpha_1 + V \sin \alpha] = R \quad \dots \quad (2)$$

$$Mk^2[\omega_1 - \omega] = Fa \quad \dots \quad (3)$$

$$V_1 \sin \alpha_1 = eV \sin \alpha \quad \dots \quad (4)$$

and since the point of contact is instantaneously brought to rest

$$V_1 \cos \alpha_1 = a\omega_1 \quad \dots \quad (5)$$

From (1), (3) and (5) 
$$V_1 \cos \alpha_1 = \frac{a(aV \cos \alpha + k^2\omega)}{a^2 + k^2} \quad \dots \quad (6)$$

From (4) and (6) 
$$\tan \alpha_1 = \frac{eV(a^2 + k^2) \sin \alpha}{a(aV \cos \alpha + k^2\omega)}$$

and 
$$V_1 = \left[ e^2 V^2 \sin^2 \alpha + \frac{a^2 (aV \cos \alpha + k^2\omega)^2}{(a^2 + k^2)^2} \right]^{\frac{1}{2}}.$$

Also from (5), 
$$\omega_1 = \frac{aV \cos \alpha + k^2\omega}{a^2 + k^2}.$$

NOTE. 5. If the angular velocity  $\omega$  be in the opposite sense to that indicated in the figure, the values of  $V_1$ ,  $\omega_1$  and  $\alpha_1$  will be those obtained by changing the sign of  $\omega$  in the above results. In particular, in order that the ball should rise in a vertical direction after bouncing, the condition  $aV \cos \alpha = k^2\omega$  must be satisfied. Again if there be no spin on the ball before bouncing it will rotate with angular velocity  $\frac{aV \cos \alpha}{a^2 + k^2}$  afterwards.

### Problems IX.

1. Prove without using the equation of work and energy that the acceleration of a solid homogeneous cylinder, which rolls with its axis horizontal down a rough inclined plane of inclination  $\alpha$ , is  $\frac{2}{3}g \sin \alpha$ . Would a hollow cylinder roll faster or slower down the same plane?

2. A uniform rod of length 6 feet oscillates about a horizontal axis 1 foot from the end. If the rod makes an angle of  $60^\circ$  with the vertical when it is at the end of its swing, find its angular velocity when it is at the bottom of its swing, neglecting friction.

3. Two equal masses are attached to the ends of a light rod which is placed horizontally on a peg at distances  $a$ ,  $b$  from the ends

of the rod. Prove that the initial accelerations of the masses are respectively  $\frac{ga(a-b)}{a^2+b^2}$  and  $\frac{gb(a-b)}{a^2+b^2}$ .

4. A wooden bobbin shaped like a cotton reel has two end discs 2" in diameter and  $\frac{1}{8}$ " thick, joined by a solid cylinder  $1\frac{1}{2}$ " long and  $\frac{1}{4}$ " in diameter; the whole weighs 1 oz. A light thread is coiled on the centre piece and the free end is pulled vertically with a force of 2 oz. Find the vertical acceleration of the free end of the thread if (1) the bobbin can turn about a fixed horizontal axis, (2) the bobbin is free to move from rest.

5. Show that a homogeneous cube, of side  $2a$ , free to turn about one of its edges, which is horizontal, oscillates in the same time as a simple pendulum of length  $\frac{4}{3}\sqrt{2a}$ .

6. A uniform thin rod, mass  $m$ , is freely hinged at its upper end and makes  $n_1$  small oscillations per second. A particle is then fixed at the lower end of the rod and the number of small oscillations is then found to be  $n_2$  per second. What is the mass of the particle?

7. A mill wheel weighing one ton and having a radius of gyration of 10 feet is provided with an axle 1 foot in diameter round one end of which a rope is coiled. Supposing the system to turn without friction in horizontal bearings, find the greatest acceleration which a man weighing ten stone and standing beneath the axle could impart to the rope without being lifted off the ground.

Find also the time he would take to turn the wheel round once, starting from rest.

Could he make the wheel turn more quickly by climbing up the rope?

8. A uniform rectangular door has a breadth of 3 ft. and a mass of 50 lb. If there is a constant frictional couple at the hinges of amount 3 lb.-feet, find the initial angular velocity that must be given to the door in order that it may swing through a right angle before coming to rest; state the units in which you express the angular velocity.

9. The lock of a railway carriage door will only engage if the angular velocity of the closing door exceeds  $\omega$ . The door swings about vertical hinges and has a radius of gyration  $k$  about a vertical axis through the hinges, whilst the centre of gravity of the door is at distance  $a$  from the line of hinges. Show that if the door be initially at rest and at right angles to the side of the train, which then begins to move with uniform acceleration  $f$ , the door will not close unassisted unless  $f > \frac{1}{2}\omega^2 k^2/a$ .

10. A cylindrical cotton reel with a hole bored through the middle has fine cotton (of which the weight may be neglected) coiled round it, and when the free end of the cotton is held in the hand it falls with five-eighths the acceleration of gravity, the axis of the reel remaining horizontal. Express the pull of the cotton in terms of the weight of the reel, the angular acceleration in terms of the linear acceleration, and the radius of gyration in terms of the radius of the reel.

11. A light string passes over a pulley of radius  $a$  which turns with friction about a horizontal axis. It is found that a mass  $m$  being fastened to one end of the string and a mass  $m'$  to the other, the former when set in motion descends with uniform velocity. When a smaller mass  $m''$  is substituted for  $m'$ , then  $m$  descends with uniform acceleration  $f$ . Assuming that there is no slipping of the string on the pulley, find the value of the friction couple (supposed constant) and the moment of inertia of the pulley.

12. A wheel is kept revolving uniformly about a horizontal axis  $1\frac{1}{2}$  inches from its centre of gravity. Find the number of revolutions per minute if the pressure on the axis varies from one-half to one and a half times the weight of the wheel. (Take  $\pi^2 = 10$ .)

Show that the greatest inclination to the vertical of the pressure of the wheel on its axis is  $30^\circ$ .

13. A circular disc rolls down another perfectly rough circular disc starting from rest at the highest point. Both discs are in the same vertical plane and the lower is fixed. Find the angle which the line joining the centres makes with the vertical when the upper disc leaves the lower.

14. A circular lamina of weight  $W$  can turn freely about a point  $A$  on its circumference, the axis being perpendicular to the plane of the lamina and horizontal. It starts from rest with the diameter through  $A$  vertically above  $A$ . Show that the resultant pressure on the axis is  $\frac{\sqrt{17}}{3}W$  when that diameter is horizontal and  $\frac{11}{3}W$  when vertically below  $A$ .

15. A uniform heavy sphere 6 inches in diameter is placed inside a cylinder 1 ft. in diameter which is fixed with axis horizontal; the sphere is slightly disturbed from rest and performs small oscillations, rolling inside the cylinder. Find the periodic time in terms of  $g$ .

16. A cube, weight  $W$ , is free to rotate about one edge which is horizontal. It moves from rest with one face in a horizontal plane. Find the action on the hinge when the face makes an angle of (i.)  $30^\circ$ , (ii.)  $45^\circ$  with the horizontal.

17. A sphere, radius  $a$ , makes  $n$  small oscillations per second rolling on the lower portion of the inner surface of a fixed sphere, radius  $b$ . If the surfaces be perfectly rough and the centre of the smaller sphere keeps in the same vertical plane, express  $g$  in terms of  $a$ ,  $b$  and  $n$ .

18. One end of a string of length  $l$  is tied to a rod of length  $a$  the other to a fixed point at height  $h$  above the ground. The rod is whirled round in a vertical plane, and when it is vertically over the fixed point the string is set free. Show that the rod will be horizontal when it strikes the ground if its centre has a velocity  $\frac{1}{2}(2n + 1)\pi(2l + a)\left\{g/(2h + 2l + a)\right\}^{\frac{1}{2}}$  when set free,  $n$  being an integer.

19. An elliptic lamina oscillates about one latus rectum as a horizontal axis and it is found that the centre of oscillation lies on the other latus rectum. Prove that the eccentricity of the ellipse is  $\frac{1}{2}$ .

20. A thin circular arc of radius  $r$  oscillates about an axis through its middle point perpendicular to the plane of the arc. Show that, whatever the length of the arc, the length of the simple equivalent pendulum is  $2r$ .

21. A circular lamina oscillates about a chord which is horizontal. If the time of oscillation be a minimum, find the distance of the chord from the centre.

22. A uniform rod is placed with one extremity on a smooth horizontal plane and the other on a smooth vertical wall. It moves from rest when it makes an angle  $\alpha$  to the vertical. Show that it will leave the wall when it makes an angle  $\cos^{-1}(\frac{1}{3} \cos \alpha)$  with the vertical.

23. A uniform rod AB of length  $l$  and mass  $m$  can turn about a hinge at A while the end B rests on a smooth wedge, angle  $\alpha$  and mass  $M$ , which can slide along a smooth horizontal plane through A. The system is allowed to move from rest when the rod is inclined at angle  $\beta$  to the horizontal. Find the angular velocity of the rod and the velocity of the wedge when the rod is inclined at angle  $\theta$  to the horizontal.

24. A right cone of angle  $2\alpha$  and weight  $W$  can turn freely about an axis passing through the centre of its base and perpendicular to its axis. If the cone starts from rest with its axis horizontal, show that when the axis is vertical, the thrust on the fixed axis is

$$W \cdot \frac{6 + 3 \cos^2 \alpha}{6 - 2 \cos^2 \alpha}.$$

25. Two uniform solid spheres each of radius  $a$  and mass  $m$  are placed with their centres a distance  $d$  apart in a perfectly rough horizontal plane. If they attract one another according to the law

$\frac{km^2}{(\text{distance})^2}$  show that the velocities of their centres when they collide is  $\left[ \frac{5km(d-2a)}{14ad} \right]^{\frac{1}{2}}$ .

26. A uniform rod is held at an inclination  $\alpha$  to the horizontal with one end in contact with a horizontal table whose coefficient of friction is  $\mu$ . If it be then released, show that it will commence to slide if

$$\mu < \frac{3 \sin \alpha \cos \alpha}{1 + 3 \sin^2 \alpha}.$$

27. A sphere of radius  $a$  is projected up a plane inclined at an angle  $\alpha$  to the horizontal, with velocity  $u$  and angular velocity  $\omega$ , in the sense which would cause it to roll up. If  $u > a\omega$ , and the coefficient of friction  $> \frac{1}{2} \tan \alpha$  show that the centre of the sphere will first come to rest at the end of time  $\frac{5u + 2a\omega}{5g \sin \alpha}$ .

28. A solid sphere rolls on the inside of a fixed hollow cylinder whose axis is horizontal. The centre of the sphere keeps in the same vertical plane and there is no slipping. Show that the sphere will make complete revolutions if, when at its lowest position, the pressure on it is greater than  $\frac{84}{7}$  times its own weight.

29. A homogeneous sphere of radius  $a$  rotating about a horizontal axis is gently placed at the lowest point of a larger rough fixed bowl of radius  $b$ ; obtain equations determining the motion, and show that if the coefficient of friction is  $\frac{1}{\sqrt{2}}$ , the angular acceleration of the

line joining the centres of the sphere and the bowl varies as  $\cos \theta$ , where  $\theta$  is the inclination of the line of centres to the vertical, up to the time at which slipping ceases and rolling begins; and that hereafter the angular acceleration varies as  $\sin \theta$ .

30. A uniform vertical circular disc, of radius  $a$ , is capable of revolving about a smooth horizontal axis through its centre; a rough flexible chain whose mass is equal to that of the plate and whose length is  $2\pi a$  hangs over its rim in equilibrium. By the principle of work and energy, show that, if one end be slightly displaced, the velocity of the chain when the other end reaches the

plate is  $\sqrt{\frac{\pi g a}{6}}$ .

31. A uniform circular disc 1 ft. in diameter, of mass 7 lb., is mounted on an axis through the centre at right angles to its plane, and one end of a coiled flat spring is attached to it, the other end of the spring being held fixed. It is found that a torque of 1 lb. ft. applied to the axis is sufficient to wind the disc through one complete turn: find the time of a small oscillation, neglecting frictional resistances.

32. A uniform rod of length  $2a$  hangs freely by one end, the other being close to the ground. An angular velocity  $\omega$  is then given to the stick, and when it has turned through a right angle the fixed end is let go. Show that on first touching the ground it will be in an upright position if  $\omega^2 = \frac{g(q^2 + 3q + 3)}{2a(q + 1)}$  where  $q$  is an odd multiple of  $\frac{\pi}{2}$ .

33. A uniform square whose sides are of length  $a$  and whose mass is  $m$  is struck a blow at one corner in a direction at right angles to a side. Determine the position of the point about which the square begins to turn. If the impulse of the blow be  $J$  find the velocity of the centre of inertia of the square and also the angular velocity.

34. Two perfectly smooth spheres are placed in unstable equilibrium, one on top of the other and the lower sphere rests on a perfectly smooth table. Show that, if a small displacement of the system be given, the spheres will separate when the straight line

joining the centres makes an angle  $\theta$  with the vertical given by the equation

$$m \cos^2 \theta = (M + m)(3 \cos \theta - 2)$$

where  $M$  is the mass of the lower and  $m$  of the upper sphere.

35. A flywheel weighing 40 lb. has a radius of gyration 9 inches ; it is driven by a couple fluctuating during each revolution so that the curve connecting couple and angular position during one revolution is a triangle, the couple becoming zero once per revolution and reaching a maximum of 2 ft. lb. There is also a constant resisting couple such that the motion is the same in each revolution. If the flywheel makes 60 revolutions a minute show that the difference between the greatest and least angular velocities is approximately 5.7 per cent. of either.

36. The axle of a fly-wheel of mass  $M$  rests in V-shaped bearings, the two arms of the V being at equal angles  $\alpha$  with the vertical. If the coefficient of friction in the bearings is  $\mu$  and a mass  $m$  hangs vertically from a string wrapped round the axle, show that the greatest value of  $m$  which will not turn the wheel is

$$\frac{\mu M}{(1 + \mu^2) \sin \alpha - \mu}.$$

Find the angular acceleration of the wheel if  $m$  is greater than the value mentioned, the radius of the axle being  $a$  and the moment of inertia of the wheel about the axis being  $I$ .

37. A light inelastic string is wrapped round a circular cylinder which can turn about its horizontal axis, and a particle is attached to the free end of the string. Show that if the cylinder is retarded by a couple proportional to the square of its angular velocity, the distance through which the particle descends in a time  $t$  from rest is  $A \log \cosh \beta t$  (where  $A$  and  $\beta$  are constants).

38. A billiard ball is at rest on a table. A cue moving horizontally strikes the ball at a point in the vertical plane containing the cue and the centre of the ball. Show that if this point is at a height from the table equal to  $\cdot 7$  the diameter of the ball, it will roll without any tendency to slip. (*cf.* the height of the cushion should be  $\cdot 7$  times the diameter of the ball).

39. A uniform rod of length  $2a$  lies upon a smooth table, and it receives a blow in a direction at right angles to its length at a point distant  $b$  from the centre. Show that the ensuing motion of the rod is the same as that of a diameter of a wheel of radius  $\frac{a^2}{3b}$  rolling upon a straight line.

40. Two particles of masses  $m, m'$  connected by a light rod of length  $a + b$  are moving on a smooth horizontal plane in a direction perpendicular to the rod with uniform velocity  $v$  ; if the rod strikes a small fixed inelastic obstacle at a point whose distances from the masses are  $a, b$  respectively, prove that the measure of the blow is  $mm'(a + b)^2 v / (ma^2 + m'b^2)$ .



41. A square lamina ABCD is rotating in its own plane about the corner A. This corner is released, and at the same instant a point P on one of the sides BC or CD is fixed. Find the position of P if the square is thereby reduced to rest.

42. A plate of mass  $m$ , in the shape of an equilateral triangle of side  $2a$ , is falling with velocity  $u$ , one side being vertical. An inelastic string tied to the opposite vertex becomes tight in a vertical position. Find the impulsive tension of the string.

43. A thin rod AB is standing upright on the edge of a shelf 4 feet from the ground. It is struck a blow and strikes the ground on the end B, in a vertical position and 1 foot from a point under the edge of the shelf. If the blow is struck in a horizontal direction at right angles to the edge of the shelf, at what point on AB is the blow struck?

44. Two rods, AB and BC, of lengths  $2a$  and  $2b$  and having the same mass per unit length, are freely jointed at B and are lying in a straight line. A blow, perpendicular to AB, is then given at the end A; show that the resulting kinetic energy when the system is free is to the energy when C is fixed as  $(4a+3b)(3a+4b) : 12(a+b)^2$ .

45. A circular disc, of mass  $M$  and radius  $a$ , is rotating with uniform angular velocity  $\omega$  on a smooth plane and impinges normally with any velocity upon a rough rod, of mass  $m$ , resting on the plane. Show that the angular velocity of the disc is reduced to  $\frac{M+m}{M+3m} \omega$ .

46. Show that if a uniform sphere can roll without slipping up or down an inclined plane, then  $7\mu \geq 2 \tan \alpha$ , where  $\mu$  is the coefficient of friction and  $\alpha$  is the inclination of the plane to the horizontal. If this condition is satisfied and the sphere is projected directly up the plane with linear velocity  $u$  and backward angular velocity  $\omega$ , show that it will commence to roll upwards when pure rolling begins.

47. A uniform solid cylinder of radius  $a$  rolls down a rough inclined plane so that its axis always remains horizontal. Show that  $3\mu > \tan \alpha$ , where  $\mu$  and  $\alpha$  are defined above.

48. A sphere of radius  $a$ , whose mass-centre G is distant  $\frac{1}{2}a$  from its centre C, is placed on a rough horizontal plane and is held at rest with CG horizontal. If the sphere rolls when CG is released, prove that, when CG makes an angle  $\phi$  with the horizontal, the horizontal and vertical components of the acceleration of G are, respectively,

$$a \left\{ \left( 1 - \frac{1}{2} \sin \phi \right) \ddot{\phi} - \frac{1}{2} \cos \phi \cdot \dot{\phi}^2 \right\}$$

and

$$\frac{1}{2}a(-\cos \phi \cdot \ddot{\phi} + \sin \phi \cdot \dot{\phi}^2).$$

If the radius of gyration of the sphere about a horizontal axis through G at right angles to CG is  $\frac{3}{4}a$ , and the sphere rolls as soon as CG is released, prove that the coefficient of friction is not less than  $32/73$ .

## CHAPTER X

CONSERVATION OF MOMENTUM AND ENERGY :  
INITIAL ACCELERATIONS AND FORCES : TENDENCY TO BREAK

**66. Angular Momentum of a Particle.** A particle of mass  $m$  is moving with velocity  $v$  at any instant and  $O$  is a fixed point at a perpendicular distance  $p$  from the tangent to the path at the given instant. The moment of momentum or the angular momentum of the particle about the point  $O$  is given by  $mv p$ .

Suppose the particle to have two simultaneous velocities  $v_1, v_2, \dots$  and  $p_1, p_2, \dots$  to be the perpendicular distances from  $O$  to the lines along which these velocities are taking place. It can be proved, by exactly the same geometrical method as is employed to prove that the algebraic sum of the moments of two forces about any point equals the moment of their resultant about the same point, that the algebraic sum of the angular momenta of the particle about any point due to two simultaneous velocities equals the angular momentum of the particle about the same point due to the resultant velocity. Hence the angular momentum of the particle is  $mv_1 p_1 + mv_2 p_2 + \dots$  due regard being taken as to sign according to the direction of rotation.

**Ex. I.**  $AB$  and  $BC$  are light rigid rods of lengths  $2a$  and  $2b$ , jointed at  $B$  and a mass  $m$  is fixed at the middle point of  $BC$ . The system rotates in one plane about  $A$ . Find the angular momentum of the mass about  $A$  when the rods are inclined to a fixed line  $Ax$  at angles  $\theta$  and  $\theta + \phi$ .

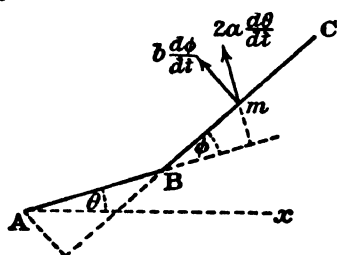


Fig. 122.

The velocity of the mass  $m$  is the resultant of the two velocities,

$2a \frac{d\theta}{dt}$  at right angles to  $AB$  and  $b \frac{d\phi}{dt}$  at right angles to  $BC$ . The angular momentum of the particle about  $A$  is therefore

$$2ma \frac{d\theta}{dt} (2a + b \cos \phi) + mb \frac{d\phi}{dt} (2a \cos \phi + b).$$

**NOTE 1.** If  $\theta$  be increasing and  $\phi$  be diminishing, the first term of the above expression will of course be positive and the second negative, for  $\frac{d\theta}{dt}$  is  $+$   $vs$  and  $\frac{d\phi}{dt}$  is  $-vs$ .

## 67. Angular Momentum of a System of Particles.

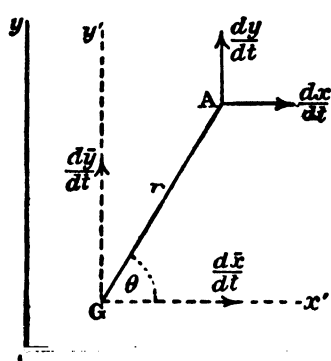


Fig. 123.

Let  $G$  be the centre of inertia of a system of particles (total mass  $M$ ) and  $A$  the position of one of the particles (mass  $m$ ). Let the co-ordinates of  $G$  and  $A$  about fixed axes  $Ox$  and  $Oy$  be  $\bar{x}$ ,  $\bar{y}$  and  $x$ ,  $y$  respectively.

The angular momentum of the mass  $m$  about  $O$  is, by the previous article, equal to

$$m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

and therefore that of the system

$$\text{is } \Sigma m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

Suppose the co-ordinates of  $A$  referred to parallel axes through  $G$  to be  $x'$ ,  $y'$ , then the angular momentum about  $O$  of the system becomes

$$\begin{aligned} & \Sigma m\left[(\bar{x} + x')\left(\frac{d\bar{y}}{dt} + \frac{dy'}{dt}\right) - (\bar{y} + y')\left(\frac{d\bar{x}}{dt} + \frac{dx'}{dt}\right)\right] \\ &= M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + \Sigma m\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right) + \frac{d\bar{y}}{dt} \cdot \Sigma mx' \\ & \quad + \bar{x} \Sigma m \frac{dy'}{dt} - \frac{d\bar{x}}{dt} \Sigma my' - \bar{y} \cdot \Sigma m \frac{dx'}{dt}. \end{aligned}$$

Now as  $G$  is the centre of inertia of the system,

$$\Sigma mx' = 0 \text{ and } \Sigma my' = 0 \text{ and hence also } \Sigma m \frac{dx'}{dt} = 0 \text{ and } \Sigma m \frac{dy'}{dt} = 0.$$

$\therefore$  the angular momentum of the system is

$$M\left(\bar{x}\frac{d\bar{y}}{dt} - \bar{y}\frac{d\bar{x}}{dt}\right) + \Sigma m\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right).$$

The first term of the above is the angular momentum about  $O$  of a particle of mass  $M$  placed at  $G$  while the second is the angular momentum of the system about its centre of inertia  $G$ .

In the case when the system of particles is a rigid body we have that if  $r$ ,  $\theta$  be the polar co-ordinates of  $A$  with respect to

the axes through G, then  $r$  is constant so that  $\frac{dr}{dt} = 0$  and  $\frac{d\theta}{dt} (= \omega)$  is the same for all particles.

$$\begin{aligned} \text{Hence} \quad & \Sigma m \left( x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) \\ &= \Sigma m \left( r \cos \theta \cdot r \cos \theta \cdot \frac{d\theta}{dt} + r \sin \theta \cdot r \sin \theta \cdot \frac{d\theta}{dt} \right) \end{aligned}$$

$= \omega \Sigma m r^2 = M k^2 \omega$ , where  $k$  is the radius of gyration of the body about an axis through G perpendicular to the plane  $xOy$ .

Thus the angular momentum of a rigid body about a fixed point O is

$$Mvp + M k^2 \omega$$

where  $p$  is the perpendicular distance from O of the direction of the velocity  $v$  of the centre of inertia.

**68. Conservation of Momentum.** Referring to the figure of the previous article, if X, Y be the external forces acting parallel to the axes on the particle at A, we have by D'Alembert's Principle that

$$\Sigma X = \Sigma m \frac{d^2x}{dt^2}, \quad \Sigma Y = \Sigma m \frac{d^2y}{dt^2} \quad \dots \quad (1)$$

$$\text{and} \quad \Sigma (xY - yX) = \Sigma m \left( x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) \quad \dots \quad (2)$$

Suppose that  $\Sigma X = 0$ , then  $\Sigma m \frac{d^2x}{dt^2} = 0$  and by integrating with respect to  $t$ , we have  $\Sigma m \frac{dx}{dt} = \text{constant}$  or  $M \frac{d\bar{x}}{dt} = \text{constant}$ .

Thus if the algebraic sum of the external forces acting on a system of particles be zero in a certain direction, then the total momentum of the system in that direction will remain constant.

This is known as the **Conservation of Linear Momentum**.

Suppose  $\Sigma (xY - yX)$ , which is the algebraic sum of the moments of the external forces about O, to be zero, then from (2)

$$\Sigma m \left( x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right) = 0$$

$$\therefore \quad \Sigma m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \text{constant}$$

or in the case of a rigid body  $Mvp + M k^2 \omega = \text{constant}$ .

Thus if the algebraic sum of the moments of the external forces acting on a system of particles, about a given point be zero, the angular momentum of the system about that point will remain constant.

This is known as the **Conservation of Angular Momentum**.

If  $X$ ,  $Y$  be impulsive forces acting for a time  $\tau$ , we have

$$\text{by integrating (1), } \Sigma \int_0^\tau X dt = \left[ \Sigma m \frac{dx}{dt} \right]_0^\tau$$

Now  $\Sigma \int_0^\tau X dt$  is the sum of the impulses of the external forces acting parallel to  $Ox$  and if this be zero  $\left[ \Sigma m \frac{dx}{dt} \right]_0^\tau$  is zero and thus the total linear momentum of the system parallel to  $Ox$  is the same after time  $\tau$  as it was initially.

Also by integrating (2), we have

$$\Sigma \int_0^\tau (xY - yX) dt = \left[ \Sigma m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right]_0^\tau$$

During the interval  $t = 0$  to  $t = \tau$ ,  $x$  and  $y$  can be considered constant so that

$$\Sigma \int_0^\tau (xY - yX) dt \text{ becomes } \Sigma \left[ x \int_0^\tau Y dt - y \int_0^\tau X dt \right]$$

which is the algebraic sum of the moments of the impulses of the external forces about  $O$ . If this be zero then

$$\left[ \Sigma m \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) \right]_0^\tau$$

is zero and thus the angular momentum of the system about  $O$  is the same after time  $\tau$  as it was initially.

Hence the principles of Conservation of Linear Momentum and Conservation of Angular Momentum apply for impulsive forces.

**Ex. II.** A circular ring of mass  $m$  and radius  $r$  lies on a smooth horizontal table. An insect of mass  $m'$  alights vertically on it and crawls round the ring with a velocity  $v$  which is uniform relative to the ring. Prove that the angular velocity of the ring is

$$\frac{m'v}{(m + 2m')r}$$

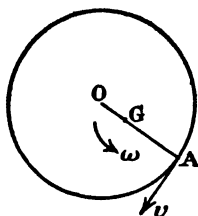


Fig. 124.

O is the centre of the ring, A the position of the insect at any instant and G the centre of inertia of the system.

The only forces acting in a horizontal plane are the reactions at A which are equal and opposite. Thus G will not move and the angular momentum about G which initially was zero will remain zero.

$$\text{Thus } m\overline{OG}^2\omega + mr^2\omega - m'\overline{AG}(v - \overline{AG}\omega) = 0$$

where  $\omega$  is the angular velocity of the ring

$$\text{i.e. } \omega \left[ m \cdot \frac{m'^2 r^2}{(m + m')^2} + mr^2 + m' \frac{m^2 r^2}{(m + m')^2} \right] = \frac{mm'rv}{m + m'}$$

$$\text{i.e. } \omega = \frac{m'v}{(m + 2m')r}$$

**Ex. III.** A uniform circular disc is spinning with uniform angular velocity  $\omega$  in its own plane about its centre. Suddenly a point in the circumference is fixed. Find the subsequent angular velocity and find the blow.

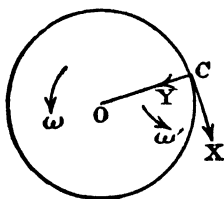


Fig. 125.

O is the centre of the disc and C the point that becomes fixed. The only forces that act are forces at C which have no moment about C. Thus equating the angular momenta about C before and after C is fixed we have

$$m \frac{a^2}{2} \omega = m \left( a^2 + \frac{a^2}{2} \right) \omega'$$

where  $m$  is the mass of the disc,  $a$  its radius, and  $\omega'$  its angular velocity after C is fixed.

$$\therefore \omega' = \frac{1}{3}\omega.$$

Let X and Y be the impulses of the forces at C perpendicular and along OC, then

$$X = ma\omega' \quad \text{and} \quad Y = 0.$$

$\therefore$  the impulse of the blow at C is  $\frac{1}{3}ma\omega$ , acting at right angles to OC.

**Ex. IV.** An inclined plane ( $\alpha$ ) of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass  $m$  is placed on its inclined face and rolls down under the action of gravity. Find the horizontal distance moved by the plane and the length of the portion rolled over by the sphere after time  $t$ .

Let  $x$  = horizontal distance moved by plane in time  $t$  and  $y$  = part of the plane rolled over by the sphere in time  $t$ .

Let R be the normal reaction on the sphere and F the friction acting on it. There will be forces of R and F acting on the plane

in opposite directions to these. Also as the horizontal plane is smooth the algebraic sum of the forces in a horizontal direction acting on the whole system is zero, hence

$$(M + m)x - my \cos \alpha = 0. \quad \dots \quad \dots \quad \dots \quad (1)$$

Considering the motion of the sphere only, we have

$$mg \sin \alpha - F = m \left( \frac{d^2 y}{dt^2} - \frac{d^2 x}{dt^2} \cos \alpha \right) \quad \dots \quad \dots \quad (2)$$

and

$$Fa = m \frac{2}{5} a^2 \cdot \frac{1}{a} \frac{d^2 y}{dt^2} \quad \dots \quad \dots \quad \dots \quad (3)$$

From (2) and (3) we have

$$\therefore \quad \frac{7}{5} \frac{d^2 y}{dt^2} - \frac{d^2 x}{dt^2} \cos \alpha = g \sin \alpha$$

$$\therefore \quad \frac{7}{5} y - x \cos \alpha = \frac{1}{2} g t^2 \sin \alpha. \quad \dots \quad \dots \quad \dots \quad (4)$$

From (1) and (4) we have

$$x = \frac{1}{2} \cdot \frac{5m \sin \alpha \cos \alpha}{7M + m(7 - 5 \cos^2 \alpha)} g t^2 \text{ and } y = \frac{1}{2} \cdot \frac{5(M + m) \sin \alpha}{7M + m(7 - 5 \cos^2 \alpha)} g t^2.$$

**69. Conservation of Energy.** Referring to the system of particles and forces dealt with in Article 25 we have

$$\Sigma X = \Sigma m \frac{d^2 x}{dt^2} = \Sigma m v_x \frac{dv_x}{dx} \quad \text{where } v_x = \frac{dx}{dt}.$$

$$\text{Also} \quad \Sigma Y = \Sigma m v_y \frac{dv_y}{dy} \quad \text{where } v_y = \frac{dy}{dt}.$$

Suppose the velocities of the typical particle whose mass is  $m$  to be  $v_{x_1}$  and  $v_{y_1}$  parallel to the axes when the co-ordinates of its position is  $x_1, y_1$ , and  $v_{x_2}$  and  $v_{y_2}$  when at the point  $x_2, y_2$ , then

$$\Sigma \left[ \int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy \right] = \frac{1}{2} \left[ \Sigma m v_{x_2}^2 \right]_{v_{x_1}} + \frac{1}{2} \left[ \Sigma m v_{y_2}^2 \right]_{v_{y_1}}$$

Thus the work done by the system of forces when the system of particles is displaced from the position determined by  $x = x_1, y = y_1$  to that determined by  $x = x_2, y = y_2$  is equal to the gain of kinetic energy provided that the system of forces is a conservative system in which case

$$\Sigma \left[ \int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy \right]$$

can be evaluated and is independent of the path taken by the particles of the system (consistent with the geometrical

configuration) during the change from the one position to the other.

Now suppose that in the position determined by  $x = a$ ,  $y = b$ , the potential energy of the system be  $C$ , then the potential energy  $P$  in any position is given by

$$P = C - \Sigma \left[ \int_a^x X dx + \int_b^y Y dy \right]$$

Let  $P_1$  and  $P_2$  be the potential energies in the positions determined by  $x = x_1, y = y_1$  and  $x = x_2, y = y_2$  respectively, then

$$P_1 = C - \Sigma \left[ \int_a^{x_1} X dx + \int_b^{y_1} Y dy \right]$$

and 
$$P_2 = C - \Sigma \left[ \int_a^{x_2} X dx + \int_b^{y_2} Y dy \right]$$

$$\begin{aligned} \therefore P_2 - P_1 &= - \left[ \int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy \right] \\ &= - \frac{1}{2} \left[ \Sigma m v_x^2 \right]_{v_{x_1}}^{v_{x_2}} - \frac{1}{2} \left[ \Sigma m v_y^2 \right]_{v_{y_1}}^{v_{y_2}} \\ &= - \frac{1}{2} \Sigma m v_2^2 + \frac{1}{2} \Sigma m v_1^2, \end{aligned}$$

where  $v_1$  and  $v_2$  are the resultant velocities of the typical particle of mass  $m$  in the positions determined by  $x = x_1, y = y_1$  and  $x = x_2, y = y_2$  respectively.

$$\text{Thus} \quad P_2 + \frac{1}{2} \Sigma m v_2^2 = P_1 + \frac{1}{2} \Sigma m v_1^2$$

and hence it follows that the sum of the potential and kinetic energies of a system of particles under the action of a conservative system of forces is constant.

This result is known as the Principle of the Conservation of Energy.

**NOTE 2.** Any system of forces which comprises forces which are explicit functions of the time or of the velocities of the particles would not be a conservative system. In particular, friction would not satisfy the condition, for its direction depends on the directions of the velocities of the particles on which it acts. The tensions of an elastic string and attractions which are entirely functions of mass and distance are examples of conservative forces.

**NOTE 3.** The proofs of the Conservation of Momentum and the Conservation of Energy given in this chapter apply for any system of particles whether rigid or not and should not be confused with the proofs of the previous chapter where consideration of a single rigid body under the action of a system of forces is dealt with.



**Ex. V.** A thin rod AB of length  $l$  and mass  $m$  has a groove along its length and is rotating with angular velocity  $\omega$  on a smooth horizontal plane about one end A which is fixed. A particle of mass  $m'$  is then placed in the groove near A: assuming there be no friction find the velocity with which the mass  $m'$  leaves the rod and the subsequent angular velocity of the latter.

The forces acting are the reaction at A and equal and opposite reactions between the particle and the groove. Hence equating the angular momenta of the system about A when the particle is at A to that when it is at B, we have

$$m\left(\frac{l^2}{4} + \frac{l^2}{12}\right)\omega = m\left(\frac{l^2}{4} + \frac{l^2}{12}\right)\omega' + m'l^2\omega'$$

where  $\omega'$  is the angular velocity of the rod when the particle has left the rod.

$$\therefore \omega' = \frac{m}{m + 3m'} \omega.$$

Also since there is no friction, the kinetic energy will remain constant, hence

$$\frac{1}{2}m\left(\frac{l^2}{4} + \frac{l^2}{12}\right)\omega^2 = \frac{1}{2}m\left(\frac{l^2}{4} + \frac{l^2}{12}\right)\omega'^2 + \frac{1}{2}m'l^2\omega'^2 + \frac{1}{2}m'v^2,$$

where  $v$  is the velocity along the rod of the particle when at B;

$$\therefore 3m'v^2 = l^2[m\omega^2 - (m + 3m')\omega'^2]$$

$$\therefore v^2 = \frac{ml^2\omega^2}{m + 3m'}.$$

The velocity of the particle after leaving the rod is therefore

$$\left[\frac{ml^2\omega^2}{m + 3m'} + \frac{m^2l^2\omega^2}{(m + 3m')^2}\right]^{\frac{1}{2}} = \frac{l\omega\sqrt{m(2m + 3m')}}{m + 3m'} \text{ in a direction}$$

inclined at  $\tan^{-1} \sqrt{\frac{m}{m + 3m'}}$  to the rod produced.

**NOTE 4.** In the case of a system of forces which is such that no work is done by the forces during any displacement of the system of particles on which they act, it follows from Article 69 that the kinetic energy of the system will be constant.

**Ex. VI.** A rod AB of mass  $m$  and length  $2a$  is attached to a smooth pivot at A. To the end B a string BC of length  $a$  is attached with a particle of mass  $m$  at C. The system lies on a smooth horizontal table and is set rotating about A with angular velocity  $\omega$ , BC being perpendicular to AB initially and fully stretched. Prove that when BC is again perpendicular to AB the angular velocity of the rod is  $\frac{13}{19}\omega$ .

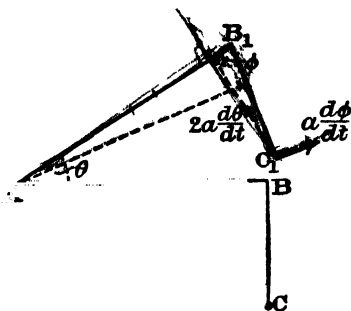


Fig. 126.

momentum, we have

$$m \left( a^2 + \frac{a^2}{3} \right) \frac{d\theta}{dt} + m \left[ 2a \frac{d\theta}{dt} (2a - a \cos \phi) + a \frac{d\phi}{dt} (a - 2a \cos \phi) \right] \\ = m \left( a^2 + \frac{a^2}{3} \right) \omega + m \cdot 4a^2 \omega$$

$$\text{i.e.,} \quad \frac{d\theta}{dt} (16 - 6 \cos \phi) + \frac{d\phi}{dt} (3 - 6 \cos \phi) = 16\omega. \quad \dots \quad (1)$$

Equating the kinetic energy of the system after time  $t$  to the initial kinetic energy, we have

$$\frac{1}{2} m \left( a^2 + \frac{a^2}{3} \right) \left( \frac{d\theta}{dt} \right)^2 + \frac{1}{2} m \left[ 4a^2 \left( \frac{d\theta}{dt} \right)^2 + a^2 \left( \frac{d\phi}{dt} \right)^2 \right. \\ \left. + 4a^2 \frac{d\theta}{dt} \cdot \frac{d\phi}{dt} \cos (180^\circ - \phi) \right] = \frac{1}{2} m \left( a^2 + \frac{a^2}{3} \right) \omega^2 + \frac{1}{2} m \cdot 4a^2 \omega^2$$

$$\text{i.e.,} \quad 16 \left( \frac{d\theta}{dt} \right)^2 + 3 \left( \frac{d\phi}{dt} \right)^2 - 12 \frac{d\theta}{dt} \cdot \frac{d\phi}{dt} \cos \phi = 16\omega^2 \quad \dots \quad (2)$$

Putting  $\phi = 90^\circ$  and eliminating  $\frac{d\phi}{dt}$  from (1) and (2), we have

$$19 \left( \frac{d\theta}{dt} \right)^2 - 32\omega \frac{d\theta}{dt} + 13\omega^2 = 0$$

$$\text{i.e.,} \quad \left( 19 \frac{d\theta}{dt} - 13\omega \right) \left( \frac{d\theta}{dt} - \omega \right) = 0.$$

$\therefore$  when BC is again perpendicular to AB, the angular velocity of the rod is  $\frac{13}{19}\omega$ .

**Ex. VII.** A, B, C are three equal particles attached to a light inextensible string at equal intervals  $a$ . The system is placed on a smooth horizontal plane with the three particles in a straight line. A blow  $P$  is applied to the middle one B in a direction perpendicular to the string. Prove that when the angle ABC is  $\theta$  the angular

velocity of AB or BC is  $P/ma(2 + \cos \theta)^{\frac{1}{2}}$ . Also find the angular acceleration of AB or BC, the acceleration of B and the tension of the string.

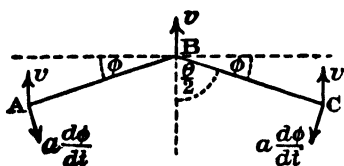


Fig. 127.

Let  $\phi$  be the angle either string makes with its original direction and  $v$  the velocity of the centre particle after time  $t$ , when  $\widehat{ABC} = \theta$ .

As the linear momentum in the direction at right angles to the original direction of the string is constant and equal to  $P$ , we have,

$$P = mv + 2m\left(v - a \frac{d\phi}{dt} \sin \frac{\theta}{2}\right) \dots \dots (1)$$

Also as the kinetic energy of the system is constant and equal to  $\frac{1}{2}m \cdot \frac{P^2}{m^2}$  (by consideration of the centre particle immediately after the blow is applied), we have

$$\frac{P^2}{2m} = \frac{1}{2}mv^2 + m\left[v^2 + a^2\left(\frac{d\phi}{dt}\right)^2 - 2va \frac{d\phi}{dt} \sin \frac{\theta}{2}\right] \dots (2)$$

Eliminating  $v$  from (1) and (2) we have

$$\frac{d\phi}{dt} = \frac{P}{am(2 + \cos \theta)^{\frac{1}{2}}}.$$

Hence 
$$\frac{d^2\phi}{dt^2} = \frac{P \sin \theta}{2am(2 + \cos \theta)^{\frac{3}{2}}} \cdot \frac{d\theta}{dt};$$

but  $\theta + 2\phi = \pi \therefore \frac{d\theta}{dt} = -2 \frac{d\phi}{dt},$

$\therefore \frac{d^2\phi}{dt^2} = -\frac{P^2 \sin \theta}{a^2 m^2 (2 + \cos \theta)^{\frac{3}{2}}}.$

Also, from (1),  $v = \frac{P}{3m} + \frac{2}{3}a \sin \frac{\theta}{2} \cdot \frac{P}{am(2 + \cos \theta)^{\frac{1}{2}}}$

$\therefore \frac{dv}{dt} = -\frac{2P^2 \cos \frac{\theta}{2}}{am^2(2 + \cos \theta)^{\frac{3}{2}}}.$

Let  $T$  be the tension of the string, then

$$-2T \cos \frac{\theta}{2} = m \frac{dv}{dt};$$

$\therefore T = \frac{P^2}{am(2 + \cos \theta)^{\frac{3}{2}}}.$

**Ex. VIII.** Two particles (masses  $m_1, m_2$ ) are connected by an elastic string of natural length  $a$ , modulus  $\lambda$ . Initially the particles are at rest on a smooth table at distance  $a$  apart. A blow  $J$  is given to the particle  $m_1$  in the direction of the string so as to stretch

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it. Prove that in the subsequent motion the greatest extension of the string is

$$\frac{J}{\sqrt{\lambda}} \sqrt{am_2/m_1(m_1 + m_2)}.$$

Let A, B be the positions of the particles initially and A<sub>1</sub>, B<sub>1</sub> the positions after a time *t*. Let A<sub>1</sub>B<sub>1</sub> = *a* + *x* and let *v*<sub>1</sub>, *v*<sub>2</sub> be the velocities of the particles when at A<sub>1</sub>, B<sub>1</sub> respectively.

The initial velocity of the mass *m*<sub>1</sub> is  $\frac{J}{m_1}$  and its kinetic energy initially is  $\frac{1}{2} \cdot \frac{J^2}{m_1}$ . The initial velocity of the mass *m*<sub>2</sub> is zero. By the conservation of linear momentum and of energy, we have

$$m_1 v_1 + m_2 v_2 = J \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\text{and} \quad \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{\lambda x^2}{2a} = \frac{1}{2} \frac{J^2}{m_1} \quad \dots \quad \dots \quad \dots \quad (2)$$

From (2), the extension of the string is a maximum when

$$m_1 v_1^2 + m_2 v_2^2 \text{ is a minimum,}$$

$$\text{i.e. when} \quad m_1 v_1 + m_2 v_2 \frac{dv_2}{dv_1} = 0,$$

$$\text{i.e. from (1), when} \quad m_1 v_1 - m_2 v_2 \cdot \frac{m_1}{m_2} = 0,$$

$$\text{i.e. when} \quad v_1 = v_2 = \frac{J}{m_1 + m_2}.$$

$$\text{Subst. in (2), we have} \quad x = \frac{J}{\sqrt{\lambda}} \sqrt{\frac{am_2}{m_1(m_1 + m_2)}}.$$

**Ex. IX.** Referring to the previous question, show that if the particles be each of mass *m* and the blow be struck at right angles to the string the maximum extension (*x*) of the string during the subsequent motion is a root of  $\frac{x(a+x)^2}{2a+x} = \frac{aJ^2}{2\lambda m}$ .

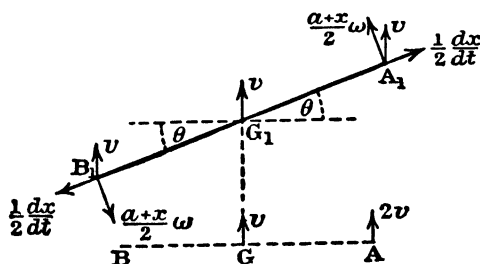


Fig. 128.

Let G be the initial position of the centre of inertia and G<sub>1</sub> its position after time *t*. Let θ be the angle the string makes with its initial position, let *x* be the extension of the string and let ω be the angular velocity of the string after time *t*.

If 2*v* be the velocity of the particle at A immediately after being struck, then

$$v = \frac{J}{2m} \quad \dots \quad \dots \quad \dots \quad (1)$$

The initial velocity of the centre of inertia will be  $v$  and, by the conservation of linear momentum, the centre of inertia will continue to move with velocity  $v$  in the direction at right angles to AB.

During the motion the angular momentum of the system about G will be constant,

$$\text{hence} \quad \frac{2m(a+x)^2\omega}{4} = m \cdot 2v \cdot \frac{a}{2},$$

$$\text{i.e.,} \quad (a+x)^2\omega = 2av. \quad \dots \quad \dots \quad \dots \quad (2)$$

Also, as the energy of the system remains constant, we have

$$\begin{aligned} & \frac{1}{2}m \left[ \left\{ v \cos \theta + \frac{a+x}{2} \omega \right\}^2 + \left\{ v \sin \theta + \frac{1}{2} \frac{dx}{dt} \right\}^2 \right] \\ & + \frac{1}{2}m \left[ \left\{ v \cos \theta - \frac{a+x}{2} \omega \right\}^2 + \left\{ v \sin \theta - \frac{1}{2} \frac{dx}{dt} \right\}^2 \right] \\ & + \frac{\lambda x^2}{2a} = \frac{1}{2}m \cdot 4v^2, \end{aligned}$$

$$\text{i.e.,} \quad (a+x)^2\omega^2 + \left( \frac{dx}{dt} \right)^2 + \frac{2\lambda x^2}{am} = 4v^2. \quad \dots \quad \dots \quad (3)$$

When the string is at its maximum extension,  $\frac{dx}{dt} = 0$ ,

$\therefore$  the maximum extension  $x$  is a root of

$$\frac{a^2 J^2}{m^2(a+x)^2} + \frac{2\lambda x^2}{am} = \frac{J^2}{m^2},$$

$$\text{i.e., of} \quad \frac{x(a+x)^2}{2a+x} = \frac{aJ^2}{2\lambda m} \quad \text{by removing the root } x=0.$$

**70. Initial Accelerations of and Forces on a System of Particles Moving from Rest.** The radial acceleration of any

particle being  $\frac{v^2}{\rho}$ , it follows that when the velocity is zero

there is no radial acceleration. The initial forces can therefore be determined without the general equations of motion being found by omitting accelerations initially zero.

**Ex. X.** A uniform rod AB is freely hinged at A and is kept in a horizontal position by means of a string. If the string be cut, determine the initial angular acceleration of the rod and the reaction at A.

Let  $2a$  be the length of the rod and  $M$  its mass.

Let  $\frac{d\omega}{dt}$  be the initial angular acceleration of the rod and  $X$  and  $Y$  the horizontal and vertical components of the reaction at A, then

$$X = 0 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$Mg - Y = Ma \frac{d\omega}{dt} \quad \dots \quad \dots \quad \dots \quad (2)$$

and 
$$Mga = M \left( a^2 + \frac{a^2}{3} \right) \frac{d\omega}{dt} \quad \dots \quad \dots \quad \dots \quad (3)$$

From (3)  $\frac{d\omega}{dt} = \frac{3g}{4a}$  and from (2),  $Y = \frac{Mg}{4}$ .

**Ex. XI.** Two uniform rods AB and CD of lengths  $2a$  and  $2b$  respectively and each of mass  $m$  per unit length are in equilibrium, the whole being hinged freely at O the middle point of AB and both rods are horizontal, CD being suspended by two light inelastic strings AC and BD each of length  $c$ . If the string AC be cut, find the initial tension of the other.

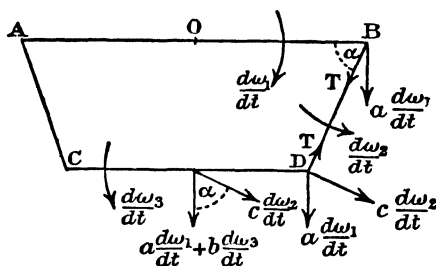


Fig. 129.

Let  $\frac{d\omega_1}{dt}$ ,  $\frac{d\omega_2}{dt}$  and  $\frac{d\omega_3}{dt}$  be the initial angular acceleration of AB, BD, and DC, and let  $\angle ABD = \alpha$  so that

$$\cos \alpha = \frac{a - b}{c}.$$

Let  $T$  be the initial tension of the string BD.

Considering the rod AB, we have

$$Ta \sin \alpha = 2am \frac{a^2}{3} \cdot \frac{d\omega_1}{dt} \quad \dots \quad \dots \quad \dots \quad (1)$$

Considering the rod CD, we have

$$T \cos \alpha = 2bmc \frac{d\omega_2}{dt} \sin \alpha \quad \dots \quad \dots \quad \dots \quad (2)$$

$$2bmg - T \sin \alpha = 2bm \left\{ a \frac{d\omega_1}{dt} + b \frac{d\omega_3}{dt} + c \frac{d\omega_2}{dt} \cos \alpha \right\} \quad \dots \quad (3)$$

and 
$$Tb \sin \alpha = 2bm \cdot \frac{b^2}{3} \frac{d\omega_3}{dt} \quad \dots \quad \dots \quad \dots \quad (4)$$

Subst. the values of  $\frac{d\omega_1}{dt}$ ,  $\frac{d\omega_2}{dt}$  and  $\frac{d\omega_3}{dt}$  from (1), (2), (4) in (3) we have

$$2bmg = T \left[ \sin \alpha + \frac{3b \sin \alpha}{a} + 3 \sin \alpha + \frac{\cos^2 \alpha}{\sin \alpha} \right].$$

$$\begin{aligned} T &= \frac{2abmg \sin \alpha}{(4a + 3b) \sin^2 \alpha + a \cos^2 \alpha} \\ &= \frac{2abc \sqrt{(c + a - b)(c - a + b)}}{ac^2 + 3(a + b)(c + a - b)(c - a + b)} mg. \end{aligned}$$

**71. Tendency of a Rigid Rod to Break on Account of its Motion.** Suppose a rigid rod AB of small cross section to be in motion under the action of external forces. The internal action at a point C of the rod consists of a tension  $T$  along the tangent to the rod at C, a shear  $S$  perpendicular to this tangent and a couple  $L$  called the stress-couple or bending moment. If the external forces acting on the portion CB be known, then as the motion of CB is known from the equations of rigid dynamics applied to the whole rod, the three equations of rigid dynamics applied to the portion CB will determine the values of  $T$ ,  $S$  and  $L$ . The value of  $L$  measures the tendency of the rod to break.

**Ex. XII.** A uniform rod having its lower end freely hinged, falls from the vertical position. Find the position at which the rod has the greatest tendency to break.

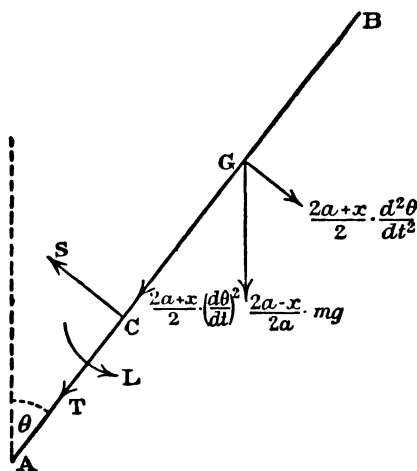


Fig. 130.

Let  $2a$  be the length of the rod and  $m$  its mass.

Let the internal action at C, where  $AC = x$ , consist of forces  $T$ ,  $S$  and a couple  $L$ .  $G$  is the middle point of  $CB$ , so that  $AG = \frac{2a+x}{2}$ .

Let  $\theta$  be the angle the rod makes with the vertical after time  $t$ , then, considering the whole rod, we have

$$m \left( a^2 + \frac{a^2}{3} \right) \frac{d^2 \theta}{dt^2} = m g a \sin \theta$$

$$\text{i.e. } \frac{d^2 \theta}{dt^2} = \frac{3g}{4a} \sin \theta. \quad \dots (1)$$

Considering the portion CB, we have

$$m \frac{2a-x}{2a} \cdot \frac{2a+x}{2} \left( \frac{d\theta}{dt} \right)^2 = m g \frac{2a-x}{2a} \cos \theta + T \quad (2)$$

$$m \frac{2a-x}{2a} \cdot \frac{2a+x}{2} \frac{d^2 \theta}{dt^2} = m g \frac{2a-x}{2a} \sin \theta - S \quad (3)$$

$$\begin{aligned} \text{and } m \frac{2a-x}{2a} \left[ \frac{(2a-x)^2}{12} + \frac{2a+x}{2} \cdot \frac{2a-x}{2} \right] \frac{d^2 \theta}{dt^2} \\ = m g \frac{2a-x}{2a} \cdot \frac{2a-x}{2} \sin \theta - L. \quad (4) \end{aligned}$$

From (4) and (1),

$$L = -m g \frac{(2a-x)^2}{4a} \sin \theta \left[ \left( \frac{2a-x}{6} + \frac{2a+x}{2} \right) \frac{3}{4a} - 1 \right]$$

$$= - \frac{x(2a - x)^2}{16a^3} mg \sin \theta.$$

The tendency to break is therefore greatest when the rod is horizontal. The position at which the rod has the greatest tendency to break is determined by the value of  $x$  which makes  $x(2a - x)^2$  a maximum.

i.e. the value of  $x$  which makes  $4a^2 - 8ax + 3x^2 = 0$

i.e. the value of  $x$  which makes  $(2a - x)(2a - 3x) = 0$ .

Now when  $x = 2a$ ,  $L$  is zero (a minimum) and therefore the position of breaking is at a point a distance  $\frac{2a}{3}$  from A.

NOTE 5. The second term in the bracket of equation (4) is obtained as follows:—The velocity of G is  $\frac{2a + x}{2} \cdot \frac{d\theta}{dt}$  and the angular momentum of CB

about C due to the motion of translation is  $m \cdot \frac{2a - x}{2a} \cdot \frac{2a + x}{2} \frac{d\theta}{dt} \frac{2a - x}{2}$ .

The rate of change of angular momentum about C due to the motion of translation is therefore  $m \frac{(2a - x)^2(2a + x)}{8a} \frac{d^2\theta}{dt^2}$ .

### Problems X.

1. A uniform square lamina moves along a straight line which coincides with one side, with a uniform velocity  $v$ . Suddenly one of the foremost corners is fixed: find the subsequent angular velocity about this point.

2. A uniform bar, AB, is rotating in a smooth horizontal plane about its centre with angular velocity  $\omega$ ; if suddenly a point P in the bar is fixed, find the position of P for which the new angular velocity will be  $\frac{1}{2}\omega$ .

3. A cube of 10 cm. edge is sliding down a smooth inclined plane of  $30^\circ$  slope, when the middle of the lower front edge meets a small obstacle. How far has it slid, if the obstruction just causes it to topple over?

4. Two equal particles connected by a light string rest with the string taut on a smooth horizontal plane. If one of the particles is struck a blow at right angles to the string, determine the paths of the particles.

5. The ends of an elastic string, of natural length  $l$  and modulus of elasticity  $\lambda$ , are fixed to two points on a smooth table at a distance  $l$  apart, and a particle of mass  $m$  is attached to its middle point. The particle is struck by a blow, of impulse  $P$ , in a direction perpendicular to the string. Show that in the ensuing motion the

string stretches by a length  $P \sqrt{\frac{l}{m\lambda}}$ .

6. Three particles A, B, C, each of mass  $m$  are rigidly connected by fine wires of equal length so as to form an equilateral triangle which rests upon a smooth table. If A receives an impulse  $I$



in the direction AB, prove that the total kinetic energy of the three particles generated by the impulse is  $\frac{5}{24} \frac{I^2}{m}$ .

7. Three particles A, B, C, of equal masses are connected by equal straight strings AB, BC and rest on a smooth horizontal plane, the angle ABC being  $135^\circ$ . If the mass C be given a blow P in a direction parallel to AB, show that the initial velocity of the mass A is  $\frac{P}{7m}$ .

8. A rigid light rod carries three masses, whose magnitudes are 1, 1, 4 at its ends and at its middle point, and is turning with angular velocity  $\Omega$  about its middle point, when one of its ends is suddenly fixed. Find the angular velocity with which it begins to turn about that end.

9. Four particles each of mass  $m$  are attached at equal intervals to an endless string and when laid on a smooth horizontal table with the string taut, they lie at the angular points of a rhombus ABCD, of which the angle at A is acute and equal to  $2\alpha$ . A blow P is given to the particle at A in the direction CA; prove that this particle begins to move with velocity  $P(1 + 2 \sin^2 \alpha)/4m$ .

10. Three equal particles A, B, C of mass  $m$  are fixed at the vertices of a weightless equilateral triangle lying on a smooth horizontal table. The particle A is given a blow in a horizontal direction perpendicular to AC; show that the initial velocities of B and C are in the ratio  $\sqrt{7} : 1$ .

11. Two equal light rods of length  $l$  are jointed freely to each other and have particles of equal weight attached to their free ends. The rods are placed symmetrically across a smooth circular cylinder of radius  $a$  with its axis horizontal so as to touch the cylinder and with the joint vertically above the axis. The system is released from rest when  $2\alpha$  is the angle between the rods. Prove that if  $2\beta$  is the angle between the rods when the system next comes to rest

$$\sin \alpha \sin \beta \tan \frac{1}{2}(\alpha + \beta) = a/l.$$

12. Four particles each of mass  $m$  are attached to the corners A, B, C, D of a rhombus formed of a light string. The whole is laid on a smooth horizontal table with the strings taut and the angle BAD acute and equal to  $2\alpha$ . A blow P is applied to the mass at A in the direction CA. Prove that the kinetic energy produced by the blow is

$$P^2(1 + 2 \sin^2 \alpha)/8m.$$

13. A plank of mass  $M$  and length  $l$  is initially at rest along a line of greatest slope of a fixed smooth plane inclined at angle  $\alpha$  to the horizontal and a man of mass  $M_1$  starting from the upper end walks down the plank so that it does not move. Show that he reaches the

other end in time  $\left[ \frac{2M_1 l}{(M + M_1) g \sin \alpha} \right]^{\frac{1}{2}}$ .

14. Two particles of masses  $M, m$  are connected by an inextensible string, and lie on a smooth table with the string fully extended. A particle of mass  $m'$  impinges directly on  $m$  with velocity  $V$  in a direction making an angle  $\alpha$  with the string. Prove that the direction of the velocity of  $m$  after impact makes an angle

$$\tan^{-1}\left(1 + \frac{M}{m}\right) \tan \alpha$$

with the string and that the velocity of  $M$  is

$$mm'V \cos \alpha / [m(m + m') + M(m + m' \sin^2 \alpha)],$$

assuming that  $m, m'$  are inelastic.

15. Three particles, each of mass  $m$ , are attached to a light string at the points  $A, B, C$ . They are placed on a smooth horizontal plane so that the angle  $ABC$  is obtuse and the portions  $AB$  and  $BC$  of the string are taut. A blow  $P$  is given to the particle  $B$  in a direction perpendicular to  $AC$ . Prove that the particle  $A$  begins to move with a velocity

$$\frac{P}{m} \cdot \frac{2 \sin A - \cos B \sin C}{3 + \sin^2 B}.$$

16. A light elastic string of natural length  $l$  and modulus  $\lambda$  is lying just stretched on a smooth horizontal table. One end is attached to a point of the table, and the other is attached to a particle of mass  $m$ . A blow of amount  $2\sqrt{m\lambda l}/3$  is applied to  $m$  perpendicularly to the string. Find equations to represent the motion of the particle, and show that in the subsequent motion, the greatest length of the string is  $2l$ .

17. Two strings of equal length have each an extremity tied to a heavy particle and their other extremities tied to two points in the same horizontal line. If one string be cut, prove that the tension of the other will be instantaneously altered in the ratio  $1 : 2 \cos^2 \alpha$ , where  $\alpha$  is the angle each string makes with the vertical.

18. A horizontal rod of mass  $m$  and length  $2a$  hangs by two parallel strings of length  $2a$  attached to its end. An angular velocity  $\omega$  is suddenly imparted to it about a vertical axis through its centre; show that the initial increase of tension of either string equals  $\frac{1}{4}ma\omega^2$ , and that the rod rises through a distance  $a^2\omega^2/6g$ .

19. A uniform rod of mass  $m$  and length  $2a$  has masses equal to  $m$  attached to each end. A string, one end of which is fastened to the middle point of the rod passes over a smooth pulley and sustains at its other end a mass  $3m$ . The system is in equilibrium with the rod horizontal. Show that when one of the particles of mass  $m$  falls off from the end of the rod, the initial acceleration of the mass  $3m$  equals  $\frac{7}{17}g$  and the initial angular acceleration of the rod is  $\frac{18g}{17a}$ .

20. A uniform rod of mass  $m$  and length  $2l$  is free to turn about a horizontal axis through its centre. Two particles each of mass  $M$  are suspended by light strings one from each end of the rod and initially the rod is at rest inclined at angle  $\alpha$  to the horizontal. If

one string be cut, find the initial tension of the other and the initial angular acceleration of the rod.

21. A uniform rod, length  $2a$  and mass  $M$  is suspended in a horizontal position by two strings attached one at each end of the rod and each is inclined at an angle  $\alpha$  with the vertical. If one string be cut, find the initial tension of the other whose length is  $l$ .

22. A rhombus ABCD consists of four smoothly jointed rods of length  $2a$  and mass  $m$ . The rhombus is suspended at A from a fixed point and closed up, so that C is coincident with A (being connected by a short string) and the rods are horizontal. The string AC is then cut. Show that the tension in the supporting string is suddenly reduced to one-quarter of itself.

23. A cube, of side  $l$ , rests with one edge on a rough plane and the opposite edge vertically over the first. It falls over and hits the plane; show that it will start rotating about another edge and that its centre of inertia will rise to a height  $\frac{l}{32}(15 + \sqrt{2})$ .

24. A uniform square lamina ABCD is suspended by vertical strings attached to A and B so that AB is horizontal. Show that, if one of the strings is cut, the tension of the other is instantly altered in the ratio 5 : 4.

25. A smooth hemispherical shell, of weight  $W$ , is held with its base against a smooth vertical wall and its lowest point on a smooth floor. The shell is then suddenly released. Show that the initial thrusts on the wall and floor are respectively  $\frac{3W}{10}$  and  $\frac{17W}{20}$ .

26. A thin uniform straight rod, of length  $l$ , can turn about one end which is fixed and is struck a blow at a distance  $\frac{3}{8}l$  from the fixed end. Show that it will be most likely to break at a distance  $\frac{\sqrt{5}a}{5}$  from the fixed end.

## CHAPTER XI

### HYDROSTATICS I

**72. A Perfect Fluid.** Whenever any forces act on a body, a state of strain is produced in the body. In the case of a flexible string, this force must act along its length, and the strain produced is necessarily an elongation. The stress at any point of the string cannot therefore have any component in a plane perpendicular to the length of the string. In the case of a rigid body, however, the stresses acting may have components normal to any plane drawn in the solid.

We have now to consider a material in which the stresses at any point, due to the action of any external forces, always act normally to any and every plane drawn through that point. Such a material is called a perfect fluid.

It follows from the definition that the fluid can offer no resistance to any force which tends to separate its particles; that is, the perfect fluid has no "viscosity."

The stress in a fluid is in all ordinary circumstances, pressure.

**73. Measurement of Pressure at a Point in a Fluid at Rest.** Taking any small area  $\delta A$  about a given point in the fluid, and denoting the pressure of the fluid on the area by  $\delta P$ , we see

that the average pressure per unit area over the area is  $\frac{\delta P}{\delta A}$ .

The pressure is not, in general, uniform over the area, but if  $\delta A$  be indefinitely diminished, and if  $\frac{\delta P}{\delta A}$  tend to the limit  $p$

then  $p$  is defined as the rate or intensity of stress at the point under consideration or, simply, the pressure at the point. Intensity of stress then is expressed in pounds per square foot, or grammes per square centimetre, or, generally, in units of force per unit area.

**74. The Pressure at any Point of a Fluid at Rest is the Same in all Directions about the Point.** Imagine the point to be within a small triangular prism of the fluid.

Let  $BCA$ ,  $B'C'A'$  be the triangular ends of the prism, and further, suppose that angles  $BCA$ ,  $B'C'A'$  are right angles.

Let  $a$ ,  $b$ ,  $c$  be the sides of the triangle, and  $l$ , the length of

the prism. Then the small prism is kept in equilibrium

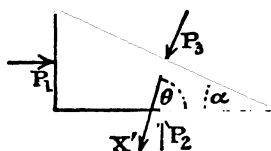
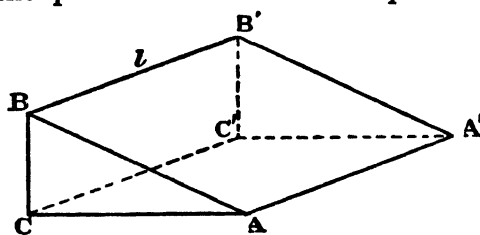


Fig. 131.

by the five pressures on its plane surfaces and the resultant external force on its mass (for instance, the weight of the fluid).

Let  $p_1$ ,  $p_2$  and  $p_3$  be the intensities of thrust on the plane faces  $BB'C'C$ ,  $CC'A'A$  and  $AA'B'B$  respectively. As these faces are indefinitely small it may be assumed that they are uniformly

pressed, so that the resultant pressure on each face will act at its middle point.

Let  $X$  be the resultant external force per unit volume, and  $X'$  its component in the middle section of the prism making an angle  $\theta$  with the side  $b$ .  $X' = kX$  where  $k$  is a proper fraction.

Resolving parallel to the face  $CC'A'A$ , we have

$$p_1 la = p_3 lc \sin \alpha + \frac{1}{2} ablkX \cos \theta$$

$$\text{or} \quad p_1 = p_3 + \frac{1}{2} bkX \cos \theta. \quad \dots \quad (1)$$

Resolving parallel to the face  $BB'C'C$ , we have

$$p_2 lb = p_3 lc \cos \alpha + \frac{1}{2} ablkX \sin \theta$$

$$\text{or} \quad p_2 = p_3 + \frac{1}{2} akX \sin \theta. \quad \dots \quad (2)$$

Hence, in the limit, when the prism becomes indefinitely small in volume, the last term in (1) and in (2) will vanish.

$$\therefore \quad p_1 = p_2 = p_3. \quad \dots \quad (3)$$

As the prism may be taken with its faces in any direction, it follows that the pressure at a point is the same in all directions about that point.

### 75. Transmission of Fluid Pressure in a Liquid at Rest.

It is a well-known experimental fact that any additional pressure applied to the surface of a liquid at rest or to any other part of the liquid is transmitted equally in all directions. This fact, however, follows readily from the definition of a perfect fluid.

Suppose that an additional pressure  $p$  per unit area is applied at a point  $P$  in a fluid, and it is required to determine the change in the pressure at a point  $Q$ . Join  $PQ$  and take

PQ as the axis of a cylinder of unit radius with its ends at P and Q perpendicular to the axis of the cylinder. Then, since the cylinder is in equilibrium, the additional pressure  $p$  at the end P must be balanced by an equal and opposite pressure  $p$  at Q. Hence the pressure at Q is also increased by an amount  $p$ . Bramah's Press is the well known example of the application of this principle.

**76. Liquids and Gases.** If a fluid is absolutely incompressible it is called a *liquid*, but the term is applied to such fluids as water, mercury, etc., which are very slightly compressible under great pressures. Gases are easily compressible, and the pressure of a gas depends upon its density and temperature.

The **elasticity** of a fluid is the quotient of a small increase in pressure by the corresponding diminution in volume per unit volume.

Thus if a change of pressure  $\delta p$  acting on a volume  $v$  causes a change of volume  $-\delta v$ ,

then 
$$\text{elasticity} = \frac{-\delta p}{\frac{\delta v}{v}} \text{ or } -v \frac{dp}{dv} \text{ in the limit.}$$

In the case of a perfect gas  $pv$  is constant, if the temperature remain constant,

$$\therefore p + v \frac{dp}{dv} = 0$$

$$\text{or} \quad -v \frac{dp}{dv} = p.$$

The elasticity of a perfect gas is therefore equal to the intensity of pressure. If the volume is measured in cubic centimetres and the pressure in dynes per sq. cm., the elasticity will be in absolute C.G.S. units.

**77. Fluid at Rest Under the Action of Gravity.** Let any point P of the fluid be at a depth  $z$  below any fixed point O in the fluid, and let Q be a point vertically beneath it at depth  $z + \delta z$ . Consider a cylinder whose axis is PQ, and whose ends are perpendicular to PQ.

Let  $a$  be the area of the section,  $p$  the intensity of pressure at P and  $p + \delta p$ , the intensity of pressure at Q.

The cylinder is acted upon in the vertical direction by an upward pressure at Q, a downward pressure at P, and its own weight. Hence, if  $\rho$  is the average density of the cylinder PQ, we have for equilibrium,

$$(p + \delta p)a - pa = g\rho a \delta z$$

where  $a$  is so small that the intensities of pressure are uniform over the area.

Hence

$$\delta p = g\rho\delta z$$

i.e.

$$\frac{dp}{dz} = g\rho. \quad \dots \quad (1)$$

(a) For a **homogeneous liquid**  $\rho$  is constant, so that the integral of equation (1) is

$$p = g\rho z + C. \quad \dots \quad (2)$$

This equation shows that if points are taken where the intensities of pressure are the same, then their depths must be the same, so that surfaces of equal pressure are horizontal planes. Hence also, the bounding surface of the liquid must be a horizontal plane.

If the fixed point  $O$  is in the free surface, and the external pressure is denoted by  $\Pi$ , then  $p = \Pi$  when  $z = 0$

$\therefore C = \Pi$ , so that equation (2) then becomes

$$p = g\rho z + \Pi. \quad \dots \quad (3)$$

Equation (3) might have been directly obtained from the equilibrium of a vertical cylinder of the liquid stretching from the surface to the point  $P$ .

From it we see that the intensity of pressure at any depth in a liquid at rest (apart from the pressure of the atmosphere) is proportional to the depth below the surface

(b) Let the fluid be a **perfect gas at uniform temperature**. Since  $p v$  is constant,  $p = k\rho$ , where  $k$  is constant. If then, we take the positive direction of  $z$  upwards, instead of downward, equation (1) will become

$$dp = -g\frac{p}{k}dz$$

or

$$\frac{dp}{p} = -\frac{g}{k}dz.$$

Integrating, and assuming  $g$  to be constant

$$\log_e \frac{p}{C} = -\frac{g}{k}z,$$

or

$$p = Ce^{-\frac{g}{k}z}. \quad \dots \quad (4)$$

The constant  $C$  must be determined from the given conditions. Thus, if the gas is air, and  $z$  is measured from the surface of the earth,  $p = \Pi$  when  $z = 0$ , therefore  $C = \Pi$  and (4) becomes

$$p = \Pi e^{-\frac{g}{k}z}. \quad \dots \quad (5)$$

If again  $\rho_0$  be the density at the surface of the earth,  
 $\Pi = k\rho_0$

$$\therefore p = \Pi e^{-\frac{g\rho_0 z}{\Pi}} \quad \dots \quad (6)$$

As in the previous case, the surfaces of equal pressure are horizontal planes.

If  $\frac{1}{k_1}$  is the co-efficient of  $z$  in equation (6),  $k_1$  is a length, and  $k_1$  and  $z$  can therefore be expressed in the same units of length.

**Ex. I.** If  $p = \Pi e^{-z/k}$ , where  $k$  is 8000 metres, find the barometric height, assuming the temperature of the air to be constant, at the top of Mount Snowdon 1088 metres, when the barometer at sea level stands at 760 mm.

Let  $h$  be the required barometric height in millimetres, and  $p$  the intensity of pressure at a height of 1088 metres. As the height of the barometer is proportional to the intensity of pressure

$$\text{then} \quad \frac{h}{760} = \frac{p}{\Pi}$$

$$\therefore \frac{h}{760} = e^{-\frac{1088}{8000}}$$

$$\therefore h = 760 e^{-\frac{1088}{8000}} \\ = 663.4 \text{ mm.}$$

**Ex. II.** In the previous exercise, find the density of the air at the earth's surface, if the density of mercury is 13.59 grams per cc.

$$\text{Here } \rho_0 = \frac{\Pi}{kg} = \frac{76 \times 13.59 \times g}{8000 \times g} \text{ grams per cc.} \\ = .00129 \text{ grams per cc.}$$

**Ex. III.** If  $\rho_0$  is .0013 grams per cc., and the density of mercury is 13.59 grams per cc., then the difference of the logs to base 10 of the heights of the barometer multiplied by 60,000 gives the approximate difference of the heights of two stations in feet.

$$\text{Here} \quad \frac{h_1}{h_2} = \frac{p_1}{p_2} = e^{\frac{g\rho_0}{\Pi}(z_2 - z_1)} \text{ from eqn. (6)}$$

Taking logs to base 10

$$\log_{10} h_1 - \log_{10} h_2 = \frac{g\rho_0}{\Pi}(z_2 - z_1) \cdot \log_{10} e$$

$$\therefore z_2 - z_1 = \frac{\Pi}{g\rho_0 \log_{10} e} \left\{ \log_{10} h_1 - \log_{10} h_2 \right\} \text{ cm.}$$

$$= \frac{76 \times 13.59 \times g}{g \times .0013 \times .4343 \times 30.48} (\log_{10} h_1 - \log_{10} h_2) \text{ ft.}$$



$$\begin{aligned}
 &= \frac{10.3284}{1.72087} \times 10^4 (\log_{10} h_1 - \log_{10} h_2) \text{feet.} \\
 &= 6.002 \times 10^4 (\log_{10} h_1 - \log_{10} h_2) \text{feet.}
 \end{aligned}$$

**78. Whole Pressure.** The whole pressure of a fluid on any surface is the sum of all the normal pressures on every element of that surface in contact with the fluid.

If  $p$  is the intensity of pressure at any point and  $\delta S$  the element of area, then the pressure on that element is  $p\delta S$ , and the whole pressure is the integral of  $p dS$  over the whole of the surface.

If the surface is a plane, then all the pressures are normal to the surface, and their resultant is the whole pressure.

**79. Resultant Pressure of a Heavy Incompressible Liquid on a Plane Area.** Let gravity be the only force acting, and let  $z$  be measured from the free surface where  $p = 0$ . Then, by Article 77,

$$p = g\rho z,$$

and resultant pressure  $= \iint p dy dz \operatorname{cosec} \alpha$ , where  $\alpha$  is the inclination of the plane area to the free surface, and  $\delta y$  is the breadth of the element of area

$$= g\rho \operatorname{cosec} \alpha \iint z dy dz.$$

Now, if  $\bar{z}$  is the depth of the centroid of the area  $S$  below the free surface

$$\text{then} \quad \bar{z}S = \operatorname{cosec} \alpha \iint z dy dz.$$

$$\therefore \quad \text{resultant pressure} = g\rho \bar{z}S. \quad \dots \quad (1)$$

If on the free surface there is an intensity of pressure (due to the atmosphere or other fluid) of  $p_0$ , then this pressure, being transmitted equally in all directions, will have a resultant pressure  $p_0 S$  on the area; and the total pressure will then be

$$(g\rho \bar{z} + p_0)S. \quad \dots \quad (2)$$

If we imagine the atmosphere or other superposed fluid to be replaced by the liquid under consideration and of such a height as to produce the same intensity of pressure at the free surface of the liquid, equation (1) will still give the resultant pressure, if by  $\bar{z}$  is now meant the depth of the centroid below the new surface (often called the "effective" surface). It is to be noted that the resultant pressure will not act at the centroid of the area but at some point below it (Art. 80).

The above argument is true whether the surface is plane or curved, for the whole pressure on any curved surface  $S$ ;

∴ whole pressure on  $S = g\rho\bar{z}S$ , where  $\bar{z}$  is the depth of the centroid of  $S$  below the effective surface.

**Ex. IV.** A semicircle is immersed vertically in a liquid with its boundary diameter in the surface. What is the resultant pressure?

In this case  $\bar{z} = \frac{4a}{3\pi}$ .

$$\begin{aligned}\therefore \text{resultant pressure} &= g\rho \cdot \frac{4a}{3\pi} \cdot \frac{\pi a^2}{2} \\ &= \frac{2}{3}g\rho a^3.\end{aligned}$$

**Ex. V.** A conical vessel of height  $h$  and vertical angle  $2\alpha$  is filled with liquid of density  $\rho$ .

Here  $S = \pi \cdot h \cdot \tan \alpha \cdot h \sec \alpha = \pi h^2 \cdot \tan \alpha \sec \alpha$  and  $\bar{z} = \frac{1}{3}h$ .

$$\begin{aligned}\text{Whole pressure on curved surface} &= g\rho \cdot \frac{1}{3}h \cdot \pi h^2 \tan \alpha \sec \alpha \\ &= \frac{\pi}{3}g\rho h^3 \tan \alpha \sec \alpha.\end{aligned}$$

**NOTE 1.** If the density varies, equation (1) does not hold; we have then to return to the equation  $\delta p = g\rho \delta z$ .

**Ex. VI.** A square of side  $a$  is immersed vertically in a liquid, the density of which varies as the depth, with one side in the surface. It is required to find the whole pressure on the square.

In this case  $\rho = kz$ , where  $k$  is a constant

$$\therefore \delta p = gkz\delta z$$

$$\therefore p = \frac{1}{2}gkz^2 + C.$$

Let  $p = 0$  when  $z = 0$  ∴  $C = 0$ .

$$\therefore p = \frac{1}{2}gkz^2.$$

Consider a horizontal strip of the square at depth  $z$  and of width  $\delta z$  then area of strip  $= a\delta z$ .

and pressure on strip  $= p a \delta z = \frac{1}{2}gkz^2 a \delta z$ .

$$\begin{aligned}\therefore \text{whole pressure on square} &= \int_0^a \frac{1}{2}gkz^2 dz \\ &= \frac{1}{6}gka^3.\end{aligned}$$

**80. Centre of Pressure.** Besides finding the resultant pressure on a plane area, we have also to find the point at which that pressure acts. This point is called the centre of pressure.

We have already proved that the intensity of pressure in a heavy liquid increases with the depth. If the area to be considered is horizontal then the pressure at every point will be the same, and the resultant pressure will act through the centroid of the area. In all other cases, the centre of pressure will be below the centroid.

Take any point  $P$  in the area, whose co-ordinates referred to fixed axes in the plane are  $x, y$ , and let  $p$  be the intensity of pressure at the point. Let  $\bar{x}, \bar{y}$  be the co-ordinates of the centre of pressure.

The pressure on an element of area is  $p\delta x\delta y$ , and its moment about the  $y$  axis is  $px\delta x\delta y$ .

The sum of the pressures is  $\iint p dx dy$ .

Taking moments, since the forces are all parallel

$$\begin{aligned} \bar{x} \iint p dx dy &= \iint p x dx dy \\ \therefore \bar{x} &= \frac{\iint p x dx dy}{\iint p dx dy} \\ \text{Similarly } \bar{y} &= \frac{\iint p y dx dy}{\iint p dx dy}, \end{aligned} \quad \left. \vphantom{\begin{aligned} \bar{x} &= \frac{\iint p x dx dy}{\iint p dx dy} \\ \bar{y} &= \frac{\iint p y dx dy}{\iint p dx dy} \end{aligned}} \right\} \quad (1)$$

the integration in both cases being taken over the whole area considered.

Note that  $\iint p dx dy = \text{resultant pressure} = g\rho\bar{z}A$ .

The formulae (i.) may be modified for special cases. If the area is in contact with a heavy homogeneous liquid then

$$p = g\rho z, ,$$

where  $z$  is the depth of the point below the free surface. It is often convenient to take as the  $x$  axis the line of intersection of the plane area with the free surface, and if  $\theta$  is the angle between the two planes

$$z = y \sin \theta, \text{ and } p = g\rho y \sin \theta.$$

The formulae (i.) then become

$$\begin{aligned} \bar{x} &= \frac{\iint xy dx dy}{\iint y dx dy} \\ \bar{y} &= \frac{\iint y^2 dx dy}{\iint y dx dy} \end{aligned} \quad \left. \vphantom{\begin{aligned} \bar{x} &= \frac{\iint xy dx dy}{\iint y dx dy} \\ \bar{y} &= \frac{\iint y^2 dx dy}{\iint y dx dy} \end{aligned}} \right\} \quad (2)$$

From these latter equations we see that the values of  $\bar{x}$  and  $\bar{y}$  remain unaltered as the plane rotates about the  $x$  axis in the surface, for the equations are independent of  $\theta$ .

The form of equations (2) suggests the following very important results:

$$\begin{aligned} \iint y^2 dx dy &\text{ is the moment of inertia of the area about } O x \\ &= A k_1^2 \text{ say, where } A \text{ is the area.} \end{aligned}$$

Suppose also that  $h$  is the  $y$  co-ordinate of the centroid.

then 
$$hA = \iint y dx dy.$$

$\therefore \bar{y} = \frac{\iint y^2 dx dy}{\iint y dx dy} = \frac{Ak_1^2}{Ah} = \frac{k_1^2}{h} \quad \dots \quad (3)$

Now let  $k$  be the radius of gyration of the plane area about an axis through its centroid parallel to the axis in the surface of the liquid.

Then

$$Ak_1^2 = Ak^2 + Ah^2$$

or 
$$\frac{k_1^2}{h} = \frac{k^2}{h} + h.$$

$\therefore \bar{y} = h + \frac{k^2}{h} \quad \dots \quad \dots \quad \dots \quad (4)$

But  $h$  is the ordinate of the centroid, therefore  $\frac{k^2}{h}$  is the distance of the centre of pressure from the centroid, measured parallel to the  $y$  axis.

Since  $\frac{k^2}{h}$  is necessarily positive it follows that the centre of pressure is always below the centroid. Also, as  $h$  becomes larger since  $k^2$  is constant,  $\frac{k^2}{h}$  diminishes, so that the centre of pressure approaches the centroid as the surface is immersed more deeply in the liquid.

The centre of pressure will always lie on the line of symmetry perpendicular to the line of intersection of the plane area with the free surface, and where it exists this line of symmetry will naturally be taken as one of the co-ordinate axes.

The following deduction from the equations (2) is also useful. Let the given area be replaced by particles  $m_1, m_2$ , etc. at points  $(x_1, y_1), (x_2, y_2)$  etc., so that the system of particles is equimomental with the given area.

$$\begin{aligned} \text{Then } \bar{x} &= \frac{\iint xy dx dy}{\iint y dx dy} & \text{and } \bar{y} &= \frac{\iint y^2 dx dy}{\iint y dx dy} \\ \text{or, by hypothesis} & & & \\ &= \frac{m_1 x_1 y_1 + m_2 x_2 y_2 + \dots}{m_1 y_1 + m_2 y_2 + \dots}, & &= \frac{m_1 y_1^2 + m_2 y_2^2 + \dots}{m_1 y_1 + m_2 y_2 + \dots} \\ &= \frac{(m_1 y_1) x_1 + (m_2 y_2) x_2 + \dots}{(m_1 y_1) + (m_2 y_2) + \dots}, & &= \frac{(m_1 y_1) y_1 + (m_2 y_2) y_2 + \dots}{(m_1 y_1) + (m_2 y_2) + \dots} \quad (5) \end{aligned}$$

But the last formulæ are those for the co-ordinates of the centre of gravity of particles  $(m_1y_1)(m_2y_2)$ , etc., with the same co-ordinates as before.

Hence if any area is equimomental to a system of particles, the centre of pressure of the area is the centre of gravity of the system of particles with their masses  $m_1m_2$ , etc., increased in the ratio of their ordinates  $y_1y_2$ , etc., i.e., increased in the ratio of their depths.

**Ex. VII.** A circular disc is immersed vertically in liquid, with its centre at depth  $d$ . Determine the depth of its centre of pressure.

Let  $a$  be the radius of the disc.

Now moment of inertia of disc about a diameter is  $\pi a^2 \cdot \frac{a^2}{4}$ , and if  $\bar{y}$  be the depth of the centre of gravity

$$\text{then, by eqn. (4), } \bar{y} = \frac{\frac{a^2}{4}}{d} + d = \frac{a^2}{4d} + d.$$

$$\text{If } d = a, \text{ then } \bar{y} = \frac{a}{4} + a = \frac{5a}{4}.$$

**Ex. VIII.** Show that the depth below the surface of a homogeneous liquid of the centre of pressure of a rectangle, two of whose sides are horizontal and at depths  $d_1$  and  $d_2$  is  $\frac{2}{3} \cdot \frac{d_1^2 + d_1d_2 + d_2^2}{d_1 + d_2}$ .

Take the line of intersection of the plane of the rectangle with the surface as the  $x$  axis, and let the area make an angle  $\theta$  with the surface.

Then, if  $\bar{y}$  is the  $y$  co-ordinate of the centre of pressure, and  $a$ , the breadth of the rectangle,

$$\begin{aligned} \bar{y} &= \frac{\iint y^2 dx dy}{\iint y dx dy} = \frac{a \int_{d_1 \csc \theta}^{d_2 \csc \theta} y^2 dy}{a \int_{d_1 \csc \theta}^{d_2 \csc \theta} y dy} \\ &= \frac{\frac{2}{3} \frac{d_2^3}{d_1^2} - \frac{d_1^3}{d_1^2} \csc \theta}{\frac{2}{3} \frac{d_2^2}{d_1^2} - \frac{d_1^2}{d_1^2} \csc \theta} = \frac{2}{3} \csc \theta \frac{d_1^2 + d_1d_2 + d_2^2}{d_1 + d_2}. \end{aligned}$$

$\therefore$  depth of the centre of pressure is  $\frac{2}{3} \frac{d_1^2 + d_1d_2 + d_2^2}{d_1 + d_2}$ .

**NOTE 2.** If the upper edge of the rectangle is in the surface,  $d_1 = 0$ , and the depth of the centre of pressure becomes  $\frac{2}{3}d_2$ . Hence the centre of pressure is at a point which is two-thirds of the width of the rectangle from its upper edge.

**Ex. IX.** If in Ex. VIII. the density of the liquid varies as the depth, find the depth of the centre of pressure.

In this case

$$\frac{dp}{dz} = g\rho \quad \text{and} \quad \rho = kz,$$

so that

$$\frac{dp}{dz} = gkz, \quad \text{and} \quad p = \frac{1}{2}gkz^2.$$

Hence, if  $\bar{z}$  is the depth of the centre of pressure

$$\begin{aligned} \bar{z} &= \frac{\int_{d_1}^{d_2} z^3 \cdot dz}{\int_{d_1}^{d_2} z^2 dz} = \frac{\frac{3}{4} d_2^4 - \frac{3}{4} d_1^4}{\frac{1}{3} d_2^3 - \frac{1}{3} d_1^3} \\ &= \frac{3}{4} \frac{(d_2 + d_1)(d_2^2 + d_1 d_2 + d_1^2)}{d_2^2 + d_1 d_2 + d_1^2}. \end{aligned}$$

**Ex. X.** A triangle is immersed in a homogeneous fluid, and its vertices are at depths  $a$ ,  $b$  and  $c$  from the surface. Show that the depth of the centre of pressure is

$$\frac{a^2 + b^2 + c^2 + bc + ca + ab}{2(a + b + c)}.$$

The depths of the middle points of the sides of the triangle are  $\frac{1}{2}(b + c)$ ,  $\frac{1}{2}(c + a)$  and  $\frac{1}{2}(a + b)$ .

Now three particles, whose masses are each proportional to one-third of the area of the triangle, placed at these middle points, will be equimomental with the triangle, and therefore the centre of pressure of the triangle will be the centre of gravity of particles whose masses are proportional to  $\frac{1}{2}(b + c)$ ,  $\frac{1}{2}(c + a)$  and  $\frac{1}{2}(a + b)$  placed at these middle points.

Take the axis of  $x$  in the surface and let  $\bar{d}$  be the depth of the centre of pressure. Then, since the depths of the middle points are proportional to their ordinates, we have, on taking moments about the axis of  $x$ ,

$$\bar{d} = \frac{\frac{1}{2} \Sigma(b + c)^2}{\frac{1}{2} \Sigma(b + c)} = \frac{a^2 + b^2 + c^2 + bc + ca + ab}{2(a + b + c)}.$$

**NOTE 3.** If one side (say BC) of the triangle be in the surface, then  $b = c = 0$  and  $\bar{d} = \frac{a}{2}$ , i.e. half the depth of the vertex.

**NOTE 4.** If one vertex (say A) of the triangle be in the surface, and BC be horizontal, then  $a = 0$ , and  $b = c$ ; hence

$$\bar{d} = \frac{2}{3}b; \quad \text{i.e., } \frac{2}{3} \text{ the depth of the horizontal side.}$$

**Ex. XI.** A trapezium is immersed in a liquid in a vertical plane with one of its parallel sides, of length  $L$ , in the surface, and the other, of length  $l$ , at a depth  $d$ . Prove that the centre of pressure is at a depth of  $\frac{d(L + 3l)}{2(L + 2l)}$ . Find also the depth of the centre of pressure if the trapezium were rotated an angle  $\alpha$  about the upper side.

Let ABCD be the trapezium, with AD in the surface.

The depth of the centre of pressure ( $d_1$ ) of triangle ABD is  $\frac{d}{2}$ .

The depth of the centre of pressure ( $d_2$ ) of triangle DBC is  $\frac{3}{4}d$ .

The resultant pressures on the triangles are proportional to the products of their areas and the depths of their centres of gravity.

If  $P_1$  and  $P_2$  be the pressures on triangles ABD, DBC respectively, then

$$\frac{P_1}{P_2} = \frac{\frac{1}{2}Ld \cdot \frac{1}{2}d}{\frac{1}{2}ld \cdot \frac{3}{4}d} = \frac{L}{2l}.$$

Therefore, if  $\bar{d}$  is the depth of the centre of pressure of the trapezium

$$\begin{aligned}\bar{d} &= \frac{P_1 d_1 + P_2 d_2}{P_1 + P_2} = \frac{L d_1 + 2l d_2}{L + 2l} = \frac{L \frac{d}{2} + 2l \cdot \frac{3}{4}d}{L + 2l} \\ &= \frac{d(L + 3l)}{2(L + 2l)}.\end{aligned}$$

Since the position of the centre of pressure, relative to the plane area, is unaltered when the area is rotated about its line of intersection with the surface, the new depth is therefore  $\bar{d} \cos \alpha$ .

**Ex. XII.** Find the position of the centre of pressure of a quadrant of a circle in a vertical plane with the bounding radius in the free surface of a homogeneous liquid.

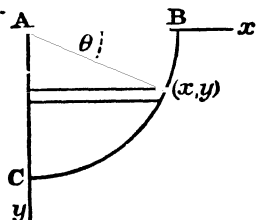


Fig. 132.

Let  $a$  be the radius of the circle and let AB be the  $x$  axis.

Take a slice of the area parallel to the free surface.

The pressure at any point of it being the same, the total pressure on the slice

$$= g\rho xy\delta y,$$

and its moment about the axis of  $x$

$$= g\rho xy^2\delta y.$$

Then, if  $\bar{y}$  is the depth of the centre of pressure,

$$\bar{y} = \frac{g\rho \int_0^a xy^2 dy}{g\rho \int_0^a xy dy}.$$

Let  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  
then

$$\bar{y} = a \cdot \frac{\int_0^{\frac{\pi}{2}} \cos^2 \theta \sin^3 \theta d\theta}{\int_0^{\frac{\pi}{2}} \sin \theta \cos^2 \theta d\theta} = a \cdot \frac{\int_0^{\frac{\pi}{2}} (\sin^2 \theta - \sin^4 \theta) d\theta}{\int_0^1 \cos^2 \theta d(\cos \theta)}$$

$$= a \cdot \frac{\frac{1}{2} \pi - \frac{3}{4} \frac{1}{2} \pi}{\frac{1}{2}} = \frac{3}{16} \pi a.$$

To determine  $\bar{x}$ , as the total pressure on the strip acts at the middle point of the strip, the moment of this pressure about the  $y$  axis =  $g\rho xy\delta y \frac{x}{2}$ .

$$\begin{aligned} \therefore \bar{x} &= \frac{g\rho \int_0^a x^2 y dy}{g\rho \int_0^a xy dy} = \frac{a \int_0^{\pi/2} \cos^3 \theta \sin \theta d\theta}{\frac{a}{2} \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta} \\ &= \frac{a \int_0^1 \cos^2 \theta d(\cos \theta)}{\frac{a}{2} \int_0^1 \cos^2 \theta d(\cos \theta)} = \frac{3}{8} a. \end{aligned}$$

**Ex. XIII.** Find the position of the centre of pressure of a semi-circular area with its bounding diameter horizontal and at a depth  $h$ .

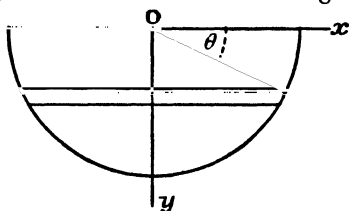


Fig. 133.

Take the origin at the centre of the circle, and measure  $y$  vertically downwards.

The total pressure on a horizontal strip at depth  $y$

$$= 2g\rho x(h + y)\delta y.$$

The moment of this pressure about the axis of  $x$  is

$$2g\rho xy(h + y)\delta y.$$

$$\therefore \text{total moment for the area} = 2g\rho \int_0^a xy(h + y)dy,$$

where  $a$  is the radius of the circle.

Let  $x = a \cos \theta$  and  $y = a \sin \theta$ , then

$$\begin{aligned} \text{moment} &= 2g\rho h \int_0^{\pi/2} a^3 \cos^2 \theta \sin \theta d\theta + 2g\rho \int_0^{\pi/2} a^4 \cos^3 \theta \sin^2 \theta d\theta \\ &= 2g\rho ha^3 \int_0^1 \cos^2 \theta d(\cos \theta) + 2g\rho a^4 \int_0^{\pi/2} \cos^2 \theta (1 - \cos^2 \theta) d\theta \\ &= \frac{2}{3} g\rho ha^3 + 2g\rho a^4 \left( \frac{1}{2} \frac{\pi}{2} - \frac{3}{4} \frac{1}{2} \frac{\pi}{2} \right) \\ &= \frac{1}{16} g\rho a^3 (16h + 3a\pi). \end{aligned}$$

But whole pressure on area = resultant pressure =  $g\rho \bar{x}A$

$$= g\rho \left( h + \frac{4a}{3\pi} \right) \frac{\pi a^2}{2}.$$



Hence, if  $\bar{y}$  is the ordinate of the centre of pressure, then

$$\begin{aligned}\bar{y} &= \frac{\frac{1}{2} g \rho a^3 (16h + 3a\pi)}{\frac{g \rho a^2}{6} (4a + 3\pi h)} \\ &= \frac{a}{4} \cdot \frac{16h + 3a\pi}{4a + 3\pi h}.\end{aligned}$$

**81. General Equations of Equilibrium.** So far, the only force which we have considered to be acting on a fluid at rest has been that of gravity.

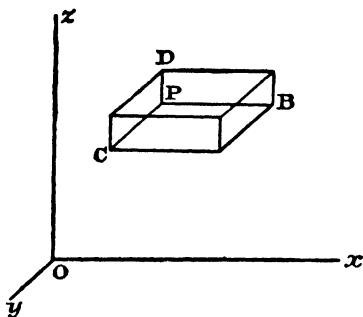


Fig. 134.

It is now necessary to take the general case of a fluid at rest under the action of any given forces.

Take three axes,  $Ox$ ,  $Oy$ ,  $Oz$  mutually at right angles and let  $P$  be any point in the liquid.

Describe about  $P$  a rectangular parallelepiped whose edges are indefinitely small and parallel to the three axes.

Let  $PB$ ,  $PC$ ,  $PD$  be three of its edges, of length  $\delta x$ ,  $\delta y$ ,  $\delta z$  respectively, let  $p$  be the intensity of pressure at  $P$  and  $\rho$  the density of the liquid at that point.

Let  $X$ ,  $Y$ ,  $Z$  be the forces per unit mass parallel to the co-ordinate axes acting on the liquid at  $P$ .

The force on and perpendicular to the face  $PCD$  is  $p \delta y \delta z$ , if we assume the face  $PCD$  to be so small that the pressure is constant over that area.

Then, if  $\frac{\partial p}{\partial x}$  is the rate at which  $p$  increases along  $Ox$ , the pressure per unit area on the surface opposite  $PCD$  is  $p + \frac{\partial p}{\partial x} \delta x$ .

$\therefore$  the whole pressure on that surface is

$$\left( p + \frac{\partial p}{\partial x} \delta x \right) \delta y \delta z$$

perpendicular to the surface.

The whole external force acting on the parallelepiped parallel to  $Ox$  is

$$X \rho \delta x \delta y \delta z$$

where  $\rho$  is the density of the fluid.

Resolving parallel to  $Ox$  we have

$$p \delta y \delta z - \left\{ p + \frac{\partial p}{\partial x} \delta x \right\} \delta y \delta z + X \rho \delta x \delta y \delta z = 0$$

$$\text{or, in the limit} \quad \frac{\partial p}{\partial x} = X \rho \quad \dots \quad (1)$$

$$\text{Similarly} \quad \frac{\partial p}{\partial y} = Y \rho \quad \dots \quad (2)$$

$$\frac{\partial p}{\partial z} = Z \rho \quad \dots \quad (3)$$

$$\text{But} \quad dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz.$$

$$\therefore \quad dp = \rho (X dx + Y dy + Z dz) \quad \dots \quad (4)$$

and this is the equation which, by integration, determines the pressure at any point.

We have already discussed the case when the only force acting is gravity. In those circumstances, taking the force of gravity parallel to  $Oz$ , we have

$$X = 0, \quad Y = 0, \quad Z = g$$

so that equation (4) becomes

$$dp = g \rho dz, \quad \text{i.e.} \quad \frac{dp}{dz} = g \rho.$$

The equations (1), (2) and (3) enable us to determine the conditions which the component forces  $X$ ,  $Y$  and  $Z$  must satisfy in order that the fluid may be in equilibrium.

$$\text{For} \quad \frac{\partial}{\partial y} (\rho X) = \frac{\partial}{\partial x} (\rho Y) \quad \text{since} \quad \frac{\partial^2 p}{\partial y \partial x} = \frac{\partial^2 p}{\partial x \partial y}$$

$$\text{Similarly} \quad \frac{\partial}{\partial z} (\rho Y) = \frac{\partial}{\partial y} (\rho Z) \quad \text{and} \quad \frac{\partial}{\partial x} (\rho Z) = \frac{\partial}{\partial z} (\rho X).$$

If we multiply the first equation by  $Z$ , the second by  $X$ , and the third by  $Y$ , we have

$$X \left( \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} \right) + Y \left( \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} \right) + Z \left( \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right) = 0$$

and this is the necessary condition for equilibrium.

**82. Elastic Fluids.** We have, when the temperature is constant

$$p = k \rho$$

$$\therefore \quad \frac{dp}{p} = \frac{1}{k} (X dx + Y dy + Z dz)$$

and if also  $X dx + Y dy + Z dz$  is a perfect differential and equal to  $-dV$

then 
$$k \frac{dp}{p} = -dV$$

or 
$$k \log \frac{p}{C} = -V.$$

$\therefore p = Ce^{-\frac{V}{k}} \quad \text{and} \quad \rho = \frac{C}{k} e^{-\frac{V}{k}}$

If the temperature is variable, then by Charles' Law and Boyle's Law in the case of a perfect gas we have

$$p = k_1 \rho \left( 1 + \frac{t}{273} \right) \\ = K\rho T, \text{ where } T \text{ is the absolute temperature} \\ \therefore \frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{KT}.$$

**83. Rotating Fluid.** A mass of homogeneous liquid, contained in a vessel in the form of a volume of revolution, revolves uniformly about a vertical axis, to determine the pressure at any point, and the surfaces of equal pressure.

Take the vertical axis as the axis of  $z$ , and measure  $z$  upwards. Let the angular velocity be  $\omega$ .

Since each particle  $m$  of the fluid is moving uniformly in a circle, it is acted on by a force  $m\omega^2 r$  towards the axis of  $z$  where  $r$  is the distance of  $m$  from that axis.

The reversed force  $m\omega^2 r$  acting outwards from the axis, together with the external force  $mg$ , will therefore reduce the particle to rest, and we can then apply equation (4) of the last article. But the components of  $m\omega^2 r$  parallel to the axes are  $m\omega^2 x$  and  $m\omega^2 y$  respectively ;

hence 
$$dp = \rho(\omega^2 x dx + \omega^2 y dy - g dz).$$

Integrating, we have

$$p = \rho \left( \frac{\omega^2 r^2}{2} - gz \right) + C, \quad \dots \quad (1)$$

where  $C$  is a constant, which may be determined from a knowledge of  $p$  at some given point.

Thus, if the origin  $O$  be taken at the point in which the free surface cuts the axis of  $z$ , so that  $r = 0, z = 0$  at  $O$  and  $p = \Pi$  (say), then  $C = \Pi$ .

$$\therefore p = \Pi + \rho \left( \frac{\omega^2 r^2}{2} - gz \right). \quad \dots \quad (2)$$

But at all points of the free surface  $p = \Pi$ , therefore the equation of this surface is

$$\frac{\omega^2 r^2}{2g} = z \quad \text{or} \quad \frac{\omega^2(x^2 + y^2)}{2g} = z.$$

Since  $z$  was measured upwards, and  $\frac{\omega^2 r^2}{2g}$  is positive, this shows that all points in the free surface are above the origin.

The equation obviously denotes a paraboloid of revolution formed by the rotation of the parabola

$$y^2 = \frac{2g}{\omega^2} z, \text{ about the } z \text{ axis.}$$

The latus rectum of the parabola is  $\frac{2g}{\omega^2}$ .

From equation (2) it also follows that all surfaces at equal pressure are paraboloids of revolution.

If the vessel contains a gas, then we have the additional condition  $p = k\rho$ ,

and equation (1) becomes

$$\frac{dp}{p} = \frac{1}{k} \left\{ \omega^2(xdx + ydy) - g dz \right\}.$$

Integrating, we have  $\log \frac{p}{A} = \frac{1}{k} \left( \frac{\omega^2 r^2}{2} - gz \right)$

so that, as before, the surfaces of equal pressures are paraboloids of revolution.

But, since  $p = k\rho$

$$\log \frac{k\rho}{A} = \frac{1}{k} \left( \frac{\omega^2 r^2}{2} - gz \right),$$

so that the surfaces of equal density are also paraboloids of revolution.

The constant  $A$  will be determined by means of some given condition, such as a knowledge of the pressure at some point, or the total mass of the fluid.

**Ex. XIV.** A closed right circular cylinder of radius  $a$ , and height  $h$  is just filled with liquid of density  $\rho$ . The cylinder and liquid then rotate with angular velocity  $\omega$  about the axis which is vertical. Find the total thrusts of the liquid on the bottom and on the top.

Taking the origin at the centre of the top surface, and measuring  $z$  downwards, we have

$$\frac{p}{\rho} = \frac{\omega^2 r^2}{2} + gz \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

there being no additive constant, since  $p = 0$  when  $r = 0$ ,  $z = 0$ .

The pressure at any point at depth  $h$  is therefore given by

$$\frac{p}{\rho} = \frac{\omega^2 r^2}{2} + gh. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

Consider an annulus on the base of radius  $r$ , and width  $\delta r$ .

Its area is  $2\pi r\delta r$ , and the intensity of pressure at any point of it is given by (2);

$$\begin{aligned}\therefore \text{ total thrust on the bottom} &= 2\pi\rho \int_0^a \left( \frac{\omega^2 r^2}{2} + gh \right) r dr \\ &= 2\pi\rho \left( \frac{\omega^2 a^4}{8} + gh \frac{a^2}{2} \right) \\ &= \frac{\pi\rho a^2}{4} \left( \omega^2 a^2 + 4gh \right)\end{aligned}$$

For the top surface  $z = 0$  and

$$\begin{aligned}\text{total thrust} &= 2\pi\rho \int_0^a \frac{\omega^2 r^3}{2} dr \\ &= \frac{1}{4}\pi\rho\omega^2 a^4.\end{aligned}$$

**Ex. XV.** A conical vessel, of height  $h$  and vertical angle  $2\alpha$  contains water whose volume is one-half that of the cone; if the vessel and the contained water revolve with uniform angular velocity, and no water overflows, show that  $\omega$  must not be greater than  $\sqrt{\frac{2g}{3h}} \cot \alpha$ .

Measuring  $z$  upwards, and taking the origin at the vertex, we obtain as the equation giving the pressure at any point,

$$p = \rho \left\{ \frac{1}{2} \omega^2 r^2 - gz \right\} + C \quad \dots \quad (1)$$

Taking the pressure at the free surface as  $\Pi$ , we have at any point in this surface,

$$\Pi = \rho \left\{ \frac{1}{2} \omega^2 r^2 - gz \right\} + C \quad \dots \quad (2)$$

If the liquid just reaches the top of the cone, then  $p = \Pi$  when  $z = h$  and  $r = h \tan \alpha$ .

$$\therefore \quad \Pi = \rho \left\{ \frac{1}{2} \omega^2 h^2 \tan^2 \alpha - gh \right\} + C \quad \dots \quad (3)$$

which gives one relation between  $\omega$  and  $C$ .

Equation (2) is the equation of a paraboloid of revolution.

Its trace on the  $zx$  plane is, from equation (2),

$$\frac{\Pi - C}{\rho} = \frac{1}{2} \omega^2 x^2 - gz.$$

i.e.

$$x^2 = \frac{2g}{\omega^2} \left( z + \frac{\Pi - C}{\rho g} \right) \quad \dots \quad (4)$$

From this equation the volume  $V_1$  of the paraboloid generated is given by

$$V_1 = \frac{\pi g}{\omega^2} \left( h + \frac{\Pi - C}{\rho g} \right)^2.$$

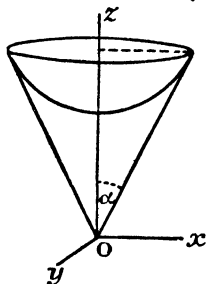


Fig. 135.

If  $V$  is the volume of the cone, then

$$V = \frac{1}{3}\pi h^3 \tan^2 a.$$

But  $V = 2V_1$

$$\therefore \frac{1}{3}\pi h^3 \tan^2 a = \frac{2\pi g}{\omega^2} \left( h + \frac{\Pi - C}{\rho g} \right)^2. \quad \dots \quad (5)$$

which gives a second relation between  $\omega$  and  $C$ .

From (3) and (5) we obtain

$$\begin{aligned} \frac{1}{3}h^3 \tan^2 a &= \frac{2g h^2}{\omega^2} \left( 1 + \frac{1}{2} \frac{\omega^2 h \tan^2 a}{g} - 1 \right)^2 \\ &= \frac{\omega^2 h^4 \tan^4 a}{2g}, \end{aligned}$$

whence

$$\omega = \sqrt{\frac{2g}{3h}} \cdot \cot a.$$

**Ex. XVI.** An open right circular cylinder of radius  $a$  and height  $h$  rotates with angular velocity  $\omega$ . Find the greatest volume of liquid it could contain assuming the liquid also to rotate with angular velocity  $\omega$  with the cylinder. Distinguish the case when the bottom is wholly covered with liquid from that when it is only partly covered.

Measuring  $z$  upwards and taking the origin at the centre of the base

$$p = \rho \left\{ \frac{1}{2} \omega^2 r^2 - gz \right\} + C. \quad \dots \quad (1)$$

At the free surface,  $p = \Pi$  and

$$\frac{\Pi - C}{\rho g} = \frac{\omega^2}{2g} (x^2 + y^2) - z. \quad \dots \quad (2)$$

The volume of the paraboloid of revolution (when the bottom is wholly covered by liquid) is, by the previous example,

$$\frac{\pi g}{\omega^2} \left( h + \frac{\Pi - C}{\rho g} \right)^2. \quad \dots \quad (3)$$

For a given velocity  $\omega$ , the greatest volume of liquid is that which just reaches the top of the cylinder.

In that case  $x^2 + y^2 = a^2$  and  $z = h$ , so that from (2)

$$h + \frac{\Pi - C}{\rho g} = \frac{\omega^2 a^2}{2g}$$

and the volume of the paraboloid is

$$\frac{\pi \omega^2 a^4}{4g}.$$

The volume of water contained

$$= \pi a^2 h - \frac{\pi \omega^2 a^4}{4g} = \frac{\pi a^2}{4g} (4gh - \omega^2 a^2)$$

Now suppose the bottom is partly uncovered.

The section of the surface by the base given by (2) where  $z = 0$ , is

$$\frac{\Pi - C}{\rho g} = \frac{\omega^2}{2g}(x^2 + y^2),$$

which is a circle of radius  $b$ , where

$$b^2 = \frac{2(\Pi - C)}{\rho \omega^2}.$$

The trace of the paraboloid on the  $xz$  plane is

$$x^2 = \frac{2g}{\omega^2}z + b^2,$$

and the volume

$$= \pi \int_0^h x^2 dz = \pi \int_0^h \left( \frac{2g}{\omega^2}z + b^2 \right) dz = \pi \left( \frac{gh^2}{\omega^2} + b^2 h \right)$$

$\therefore$  volume of water

$$= \pi a^2 h - \pi \left( \frac{g}{\omega^2} h^2 + b^2 h \right) = \pi h \left\{ (a^2 - b^2) - \frac{gh}{\omega^2} \right\}.$$

NOTE 5. The angular velocity  $\omega$  required to empty the cylinder is given by  $\omega^2 = \frac{gh}{a^2 - b^2}$ ; but in this case  $b = a$ , so that  $\omega$  must be infinitely great.

**Ex. XVII.** Liquid is contained within a thin circular tube in a vertical plane which can rotate about a vertical diameter. If the liquid subtend an angle  $2\alpha$  at the centre, prove that it will separate into two parts if the angular velocity exceeds  $\sqrt{\frac{g}{a}} \sec \frac{\alpha}{2}$ , where  $a$  is the radius of the circle.

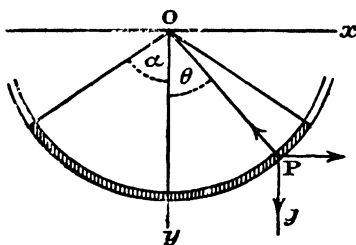


FIG. 136.

Take horizontal and vertical axes through O, the centre of the circle, and measure  $y$  downwards. Then

$$dp = \rho(\omega^2 x dx + g dy)$$

$$\therefore p = \rho \left( \frac{\omega^2 x^2}{2} + gy \right) + C. \quad (1)$$

Writing

$$x = a \sin \theta, \quad y = a \cos \theta,$$

$$p = \rho \left( \frac{\omega^2 a^2 \sin^2 \theta}{2} + ga \cos \theta \right) + C \quad \dots \quad (2)$$

$$\text{But } p = 0 \text{ when } \theta = \alpha \quad \therefore C = -\rho \left\{ \frac{\omega^2 a^2}{2} \sin^2 \alpha + ga \cos \alpha \right\}$$

$$\text{and } p = \rho \left\{ \frac{\omega^2 a^2}{2} (\sin^2 \theta - \sin^2 \alpha) + ga (\cos \theta - \cos \alpha) \right\} \quad (3)$$

If  $\theta = 0$ , we obtain the intensity of pressure ( $p_0$ ) at the lowest point of the liquid,

$$\text{i.e.,} \quad p_0 = \rho \left\{ ga(1 - \cos \alpha) - \frac{\omega^2 a^3}{2} \sin^2 \alpha \right\}.$$

The liquid will therefore separate into parts if the angular velocity is such that

$$ga(1 - \cos \alpha) - \frac{\omega^2 a^3}{2} \sin^2 \alpha$$

is negative

$$\text{i.e., if} \quad \omega^2 > \frac{2g(1 - \cos \alpha)}{a \sin^2 \alpha}$$

$$> \frac{g}{a} \sec^2 \frac{\alpha}{2}.$$

$$\text{i.e., if} \quad \omega > \sqrt{\frac{g}{a} \sec^2 \frac{\alpha}{2}}$$

### Problems XI.

1. The pressure and density of a certain point on the earth's surface are  $p_0$  and  $\rho_0$  respectively. Assuming that the atmosphere obeys Boyle's Law, find the pressure at height  $h$ .

2. Assuming that the temperature in the atmosphere is constant, prove that the relation between the pressure  $p$ , and the height  $z$  above sea-level is  $p = p_0 e^{-z/h}$ , where  $p_0$  is the pressure at sea level.

Taking  $\log_{10} e = .4343$  and the constant  $h$  to be 8000 metres, calculate the barometric height at the top of the Jungfrau, 4160 metres high, when the barometer stands at 760 mm. at sea level.

3. The section of a dam is a right-angled triangle with the water-side vertical and the hypotenuse inclined to the vertical at an angle  $\alpha$ . The dam is constructed of masonry of specific gravity 4.5. Show that in order that the weight of the dam may be sufficient to prevent the water from overturning it,  $\tan \alpha$  must be greater than  $\frac{1}{3}$ , and assuming the masonry merely to rest on rough ground and to be prevented from sliding backwards by friction (coefficient of friction  $\mu$ ) show that  $\tan \alpha$  must also be greater than  $\frac{2}{9\mu}$ . What

would be the minimum values of  $\tan \alpha$ , if the depth of the water, at its highest level, was three-quarters the height of the dam?

4. A trap-door in the side of a water-tank is 2 feet square and is kept in place by forces applied at the top and bottom edges. Calculate the magnitudes of these forces, when the surface of the water is 4 feet above the upper edge of the trap door, assuming that 36 cubic feet of water weigh a ton.

5. A layer of liquid of depth 21 inches rests upon a liquid of three times its density. A rectangle is immersed in both liquids with one pair of sides, of length 10 inches, vertical. Find the lengths into which the vertical sides are divided by the boundary between the liquids if the thrusts on the two portions are equal.



6. Each gate of a lock is 20 feet high, and 6 feet wide, and is supported on pivots, situated 2 feet from the top and bottom. The angle between the gates when they are closed is  $140^\circ$ . If the depths of water on the two sides are 17 feet and 5 feet respectively, find the magnitude and position of the resultant water pressure on each gate, the magnitude of the reaction between the gates and the magnitude and directions of the reactions at the pivots, on the assumption that the gate reaction acts in the same horizontal plane as the resultant water pressure.

7. A triangle is totally immersed in water with its base horizontal. Show that the depth of its centre of pressure is the same as that of the centre of gravity of particles placed at the middle points of its sides, whose masses are proportional to the depths of these middle points.

8. Prove that the centre of pressure of a triangular area occupying any position in a fluid is the centre of gravity of three particles placed at the middle points of the sides, their masses being proportional to their depths.

ABCD is a trapezium whose sides have the lengths :  $AB = 24$ ,  $BC = 13$ ,  $CD = 10$ ,  $DA = 15$  ; the parallel sides are AB and CD ; the trapezium is placed with A in the surface of water and AB vertically down. Prove that the depth of the centre of pressure is 15.08, and that its distance from the line joining the mid-points of AD and BC is 0.68.

9. A vertical rectangular dock gate has a depth of 30 feet and a breadth of 40 feet. On one side the depth of the water is 24 feet and on the other side it is 6 feet. Find, by integration, the resultant thrust on the gate and the point where it acts. Take the sp. gr. of the water to be 1.025.

10. ABCD is a trapezium in which AB is 8, CD 12, and AD and BC each 5 feet. It is placed vertically in water with AB in the surface. Find how far it must be lowered from this position, still being vertical, to make the distance from the centre of gravity and the centre of pressure less than 1 inch. Neglect atmospheric pressure.

11. A square whose sides are of length  $a$  is immersed in a liquid in a vertical plane with one corner in the surface and with one diagonal vertical. Find the depth of the centre of pressure.

12. A triangle of height 4 feet is immersed in a liquid with its base parallel to the surface and at a depth 3 feet, the opposite vertex being at a depth of 6 feet. Show that the centre of pressure is at a distance of 2 inches from the centre of gravity.

13. A triangle of base  $a$  and altitude  $h$  is immersed in a liquid with its base horizontal, its vertex uppermost, and at a depth  $k$ . Find the depth of the centre of pressure. What would the depth of the vertex have to be in order that the height of the centre of pressure

above the base should be  $\frac{7h}{24}$ ?

14. Show that the depths of the centre of pressure of a lamina totally immersed in water with its plane vertical is greater than the depth  $z$  of the centre of gravity below the free surface of the water, by an amount which varies inversely as  $z$ ; atmospheric pressure is neglected.

A circular lamina of radius 1 foot is totally immersed in water with a horizontal diameter fixed at a depth of 3 feet. Show that if the lamina be rotated about this diameter, the centre of pressure always lies on a fixed vertical circle of diameter 1 inch.

15. A rhombus, immersed in water, has a vertex in the surface and the diagonal (length  $l$ ) through that vertex, vertical. Prove that the distance of the centre of pressure from this vertex is  $\frac{7l}{12}$ .

16. ABCD is a quadrilateral which has its side AB in the surface of a liquid and the sides AD and BC vertical and equal to  $a$  and  $b$  respectively. Prove that the depth of the centre of pressure is

$$\frac{(a^2 + b^2)(a + b)}{2(a^2 + ab + b^2)}.$$

17. A parallelogram has its corners at depths  $d_1, d_2, d_3, d_4$  below the surface of a liquid and its centre at a depth  $d$ . Show that the depth of the centre of pressure is

$$(d_1^2 + d_2^2 + d_3^2 + d_4^2 + 8d^2)/12d.$$

18. If  $a, b, c, d$  be the depths of the four corners of a quadrilateral immersed in water, prove that the depth of the centre of pressure is  $\frac{1}{2}\Sigma a - \frac{1}{6h}\Sigma ab$  where  $h$  is the depth of the centre of gravity.

19. A square whose side is  $2a$  is immersed with its plane vertical in water and its centre C is at depth  $h$ . Shew that, if the square be rotated about C in a vertical plane, the depth of the centre of pressure is constant and equal to  $\frac{3h^2 + a^2}{3h}$ .

20. One face of a rectangular box of depth  $d$  is hinged about its upper edge which is horizontal. A force  $F$  applied at the middle point of the lower horizontal edge and at right angles to its plane just causes the face to rotate. What is the greatest depth of liquid of density  $\rho$  that the box could hold?

21. A square whose side is of length  $a$  is immersed in a liquid, in a vertical plane and with two of its sides horizontal. It moves downwards with constant velocity  $V$ . Express the velocity of the centre of pressure in terms of its depth  $x$  below the surface.

22. A sluice-gate is of width  $a$  and depth  $b$ ; the water stands on one side of it to a depth  $h$  above its lowest edge and on the other to a depth  $h'$ ; if it be supported at both its upper and lower edges, find the necessary horizontal forces to be applied at these edges to give equilibrium.

23. A closed right circular cylinder, axis vertical, is just filled with liquid and the whole rotates about the axis of the cylinder. Find the angular velocity in order that the total thrust on the bottom shall be twice the weight of the liquid.

24. An open right conical vessel of radius  $r$  and depth  $d$  is filled with water and the whole rotates about the vertical axis. Find the greatest angular velocity  $\omega$  in order that water should be at the brim. Find the volume of water that runs out when the angular velocity is (i.)  $\frac{1}{2}\omega$ , (ii.)  $2\omega$ .

25. An open hemispherical bowl of radius  $r$  contains liquid and the whole rotates about the vertical radius with angular velocity  $\omega$ , the brim being in a horizontal plane. What is the greatest amount of water that the bowl could contain? Also find the angular velocity in order that the depth of liquid at the centre should be  $d$ .

26. A circular cylinder, closed at the top, is of height  $h$  and radius  $a$ ; it is just filled with liquid and revolves with uniform velocity  $\omega$  about its axis. If the thrust on the base be  $n$  times that on the top, show that  $\omega$  is equal to  $\frac{2}{a}\sqrt{\frac{gh}{n-1}}$ .

27. Prove that, when a liquid rotates with uniform angular velocity about a vertical axis, the surfaces of equal pressure are paraboloids of revolution.

The whole circumference and a diameter of a circle are formed by fine uniform tubes, communicating freely with one another. The diameter is vertical, and initially the tubes are filled with water to the level of the centre of the circle. Prove that all the water will be driven into the circumference, if the circle is made to rotate about its diameter with an angular velocity greater than

$$\sqrt{\left\{\frac{2g}{a(1-\sin \frac{1}{2}\theta)}\right\}},$$

where  $a$  is the radius, and  $\theta$  is the unit of circular measure.

28. Obtain the answer to Ex. XIII, page 217, by using result (4), page 213.

## CHAPTER XII

### HYDROSTATICS II

**84. Resultant Pressures on Curved Surfaces.** In determining the resultant fluid pressure on any curved surface, the resultant vertical pressure and the resultant horizontal pressures in two directions at right angles are found.

In general, these three forces cannot be compounded into a single force. To find the resultant vertical pressure due to a heavy liquid, consider the equilibrium of a vertical cylinder of the liquid of which the base is the surface considered. Resolving vertically, we see that the component of the pressure in the vertical direction must balance the weight of the cylinder of liquid.

Therefore the vertical component of the pressure is the weight of the column of liquid contained within the vertical lines drawn from the boundary of the area to the free surface and acts in the vertical line through the centre of gravity of the liquid.

To find the resultant horizontal pressure in any given direction consider the equilibrium of a cylinder of the liquid contained within horizontal lines drawn from the boundary of the surface parallel to the given direction to meet any vertical plane.

Resolving horizontally, we observe that the horizontal pressure on the vertical section must be balanced by the resultant horizontal pressure of the given surface on the liquid, as all other forces act in vertical planes parallel to the vertical section.

Hence the resultant horizontal pressure on a surface in any given direction is equal to that on its projection perpendicular to the given direction and acts through the centre of pressure of that plane area.

For let  $A$  and  $A'$  be the curved surface and its projection respectively. Consider an element  $\delta A$  of  $A$ , and let  $\delta A'$  be its corresponding projection. The elementary volume of fluid enclosed within the horizontal lines joining  $\delta A$  and  $\delta A'$  is in equilibrium under the normal pressures on its surfaces and its own weight. Resolving horizontally, we see that the horizontal pressures on  $\delta A$  and  $\delta A'$  must be equal and opposite. Hence the resultant horizontal pressures on  $A$  and  $A'$  must also be equal in magnitude and opposite in direction, so that the resultant horizontal pressure on  $A$  acts in the line through the centre of pressure of  $A$ .



Thus, let  $R$  be the resultant pressure, making an angle  $\theta$  with the axis, and suppose further that  $h = a$ .

$$\begin{aligned} \text{Then} \quad R \cos \theta &= X & \dots & \dots & \dots & (1) \\ R \sin \theta &= Z & \dots & \dots & \dots & (2) \end{aligned}$$

so that

$$R^2 = X^2 + Z^2 = \left\{ \frac{g\rho a^3}{6} (3\pi + 4) \right\}^2 + \left\{ \frac{g\rho a^3}{6} (\pi + 6) \right\}^2$$

$$\therefore R = \frac{g\rho a^3}{6} \sqrt{(10\pi^2 + 36\pi + 52)}$$

and

$$\tan \theta = \frac{Z}{X} = \frac{\pi + 6}{3\pi + 4}.$$

**Ex. II.** A vertical semicircle of radius  $a$  is immersed in water with its diameter in the surface. If the semicircle is the plane face of a spherical quadrant find the resultant pressure on the curved surface.

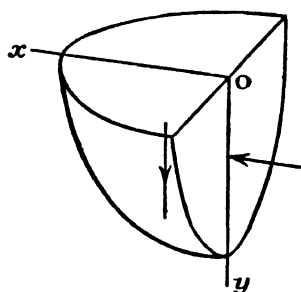


Fig. 138.

Take the centre as origin, the radius bisecting the surface area as the axis of  $x$  and the vertical radius as the axis of  $y$ .

The pressure parallel to  $Ox$  is equal to the pressure on the vertical semicircle since this area is the projection on a vertical plane of the curved surface.

This pressure is therefore

$$= g\rho \cdot \frac{4a}{3\pi} \cdot \frac{\pi a^2}{2} = \frac{2}{3} g\rho a^3.$$

and the co-ordinates of its point of action are

$$0, \frac{3}{16}\pi a \quad (\text{See Ex. XII. Art. 80.})$$

The resultant horizontal pressure on the curved surface in the direction perpendicular to  $Ox$  is evidently zero.

The resultant vertical pressure = the weight of the fluid.

$$= \frac{1}{8} g\rho \pi a^3.$$

and acts in the direction of the line  $x = \frac{3}{8}a$ .

These two forces lie in a plane, and meet in the point

$$x = \frac{3}{8}a, \quad y = \frac{3}{16}\pi a,$$

and they are therefore equivalent to the single force  $\frac{1}{8}\sqrt{\pi^2 + 4} g\rho a^3$ .

If  $\theta$  is the angle which the force makes with the  $x$  axis

$$\text{then } \tan \theta = \frac{1}{4}\pi.$$

But  $\frac{y}{x} = \frac{1}{4}\pi$  and therefore the resultant fluid pressure acts through the centre of the sphere, as is otherwise obvious, since all the fluid pressures are normal to the surface and therefore pass through the centre.

**85. Equilibrium of Floating Bodies.** If a body is floating in a heavy fluid at rest, two conditions determine its equilibrium.

1. The weight of the body must be equal to the resultant fluid pressure and this pressure is equal to the weight of fluid displaced.
2. The centre of gravity of the body must be in the same vertical line as the centre of gravity of the displaced fluid.

In the case of a solid body floating in water, or other liquid, and only partially immersed, the weight of the body is equal to the weight of liquid displaced, together with the weight of air displaced. In the exercises which follow the weight of air displaced will be neglected.

If a body floats in a homogeneous liquid, the plane in which the body is intersected by the surface is called the plane of flotation, and the centre of gravity of the displaced liquid is called the centre of buoyancy.

**Ex. III.** A square board is placed in a liquid of 4 times its own density. Show that there are three different positions in which it will float with one corner only below the surface.

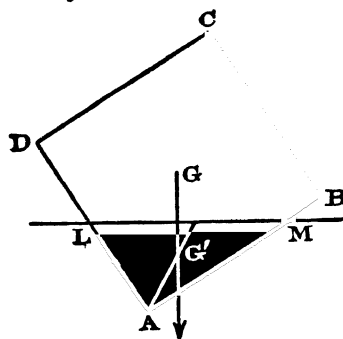


Fig. 139.

Let ABCD be the board with A in the liquid, and LM the line of intersection of the surface of the liquid with the square.

Let  $AL = b$ ,  $AM = a$ , and  $AB = c$ . Then, from the first condition, since the liquid is 4 times denser than the solid,

$$\triangle LAM = \frac{1}{4} \text{sq. } ABCD$$

$$\therefore 2ab = c^2 \dots \dots (1)$$

The second condition is that  $GG'$  shall be perpendicular to  $LM$ ,  $G$  being the centroid of the board, and  $G'$  the centroid of the triangle  $LAM$ .

If  $A$  be taken as the origin, and  $AB$ ,  $AD$  as axes, then the co-ordinates of  $G$ , are  $\frac{c}{2}, \frac{c}{2}$

and of  $G'$   $\frac{a}{3}, \frac{b}{3}$ .

and the equation of  $GG'$  is

$$x\left(\frac{c}{2} - \frac{b}{3}\right) - y\left(\frac{c}{2} - \frac{a}{3}\right) + \frac{cb}{6} - \frac{ca}{6} = 0. \dots (2)$$

But the equation of  $LM$  is

$$\frac{x}{a} + \frac{y}{b} = 1, \dots \dots \dots (3)$$

and, if the lines  $GG'$  and  $LM$  are perpendicular,

then 
$$\frac{1}{a}\left(\frac{c}{2} - \frac{b}{3}\right) - \frac{1}{b}\left(\frac{c}{2} - \frac{a}{3}\right) = 0,$$

i.e., 
$$b(3c - 2b) - a(3c - 2a) = 0,$$

i.e., 
$$2(a^2 - b^2) - 3c(a - b) = 0.$$

Hence, one solution is  $a = b$ ;  
but  $2ab = c^2$  from (1)

$\therefore a = b = \frac{1}{\sqrt{2}}c$ . and AC is vertical.

The other is given by

$$\begin{aligned} 2(a + b) &= 3c \\ \text{and } 2ab &= c^2 \end{aligned}$$

from which  $a = c$ ;  $b = \frac{c}{2}$ .

or  $a = \frac{c}{2}$ ;  $b = c$ .

so that there are three different positions of equilibrium.

**Ex. IV.** A solid body in the form of a right circular cone of height  $h$  is floating with its axis vertical and vertex downwards in a lake of water. Find the work done in pushing it down very slowly until it is just immersed.

Let  $\alpha$  be the semi-vertical angle of the cone,  $W$  its weight, and  $h$  its height;  $s$  the density of the water,  $S$  the density of the cone and  $z$  the depth of the cone in the water.

Volume of cone  $= \frac{1}{3}\pi h^3 \tan^2 \alpha$ .

Volume of water displaced  $= \frac{1}{3}\pi z^3 \tan^2 \alpha$ .

$\therefore$  weight of cone  $= W =$  weight of water displaced

$\therefore \quad \quad \quad = \frac{1}{3}\pi z^3 s \tan^2 \alpha. \quad \quad \quad \dots \quad \dots \quad (1)$

and weight of cone  $\quad \quad \quad = \frac{1}{3}\pi h^3 S \tan^2 \alpha \quad \quad \quad \dots \quad \dots \quad (2)$

$\therefore \quad \quad \quad \frac{z^3}{h^3} = \frac{S}{s}, \text{ or } \frac{z}{h} = \left(\frac{S}{s}\right)^{\frac{1}{3}}. \quad \quad \dots \quad \dots \quad (3)$

Let the cone be depressed a further depth  $x$

then resultant vertical force  $=$  weight of excess liquid displaced

$$\begin{aligned} &= \frac{1}{3}\pi s[(z + x)^3 - z^3] \tan^2 \alpha \\ &= \frac{1}{3}\pi s(3z^2x + 3zx^2 + x^3) \tan^2 \alpha. \end{aligned}$$

Then, work done during a small vertical displacement  $\delta x$  is  $\frac{1}{3}\pi s(3z^2x + 3zx^2 + x^3) \tan^2 \alpha \delta x$ .

$\therefore$  total work done during immersion

$$\begin{aligned} &= \frac{1}{3}\pi s \tan^2 \alpha \int_0^{h-s} (3z^2x + 3zx^2 + x^3) dx \\ &= \frac{1}{3}\pi s \tan^2 \alpha \left[ \frac{3}{2}z^2x^2 + zx^3 + \frac{x^4}{4} \right]_0^{h-s}. \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{12} \pi s \tan^2 \alpha (h-z)^2 [6z^2 + 4z(h-z) + (h-z)^2] \\
&= \frac{1}{12} \pi s \tan^2 \alpha (h-z)^2 (3z^2 + 2hz + h^2) \\
&= \frac{1}{12} \pi s \tan^2 \alpha h^4 \left(1 - \frac{z}{h}\right)^2 \left(3 \frac{z^2}{h^2} + 2 \frac{z}{h} + 1\right)
\end{aligned}$$

But  $\frac{z}{h} = \left(\frac{S}{s}\right)^{\frac{1}{3}} = k$  (say).

and  $W = \frac{1}{3} \pi h^3 S \tan^2 \alpha$ .

$$\begin{aligned}
\therefore \text{ total work done} &= \frac{hW}{4k^3} \left(1 - k\right)^2 \left(3k^2 + 2k + 1\right) \\
&= \frac{hW}{4} \left(3k - 4 + \frac{1}{k^3}\right).
\end{aligned}$$

**86. Body Floating Under Constraint.** If, for instance, a solid is floating in liquid, and is partly supported by a string attached to some fixed point, since the weight of the body and the fluid pressure act in the vertical direction, it is evident that the string must also be vertical, and the tension in it will be equal to the difference of the weight of the body and that of the liquid displaced.

If two or more constraining forces act, their resultant must act in the vertical direction.

**Ex. V.** A uniform rod, capable of turning about one of its ends, which is out of the water, rests inclined to the vertical. If its specific gravity is  $5/9$ , find the length immersed, the cross section being small.

Let  $2l$  be the length of the rod,  $a$  its cross-sectional area and let  $w$  be the weight of unit volume of water, then weight of rod

$$= 2law \cdot \frac{5}{9}.$$

Let  $2x$  be the length immersed. Then weight of water displaced  
 $= 2xaw.$

Taking moments about the supported end for the equilibrium of the rod, we have

$$\begin{aligned}
2law \cdot \frac{5}{9} \cdot l &= 2xaw(2l - x), & \dots & \dots & \dots & (1) \\
\text{from which} & 9x^2 - 18lx + 5l^2 = 0 \\
\text{i.e.,} & (3x - l)(3x - 5l) = 0.
\end{aligned}$$

$$\text{Hence } x = \frac{1}{3}l \text{ or } x = \frac{5}{3}l.$$

The larger value being inadmissible, one third of the rod is immersed.

**NOTE 1.** If  $\theta$  is the inclination of the rod to the horizontal then both sides of (1) contain the term  $\cos \theta$ . Hence one solution of the equation is  $\cos \theta = 0$  or  $\theta = \frac{\pi}{2}$ , so that a possible position of equilibrium is that for which the rod is vertical. If  $h$  is the height of the end above the water,

$$\sin \theta = \frac{h}{2(l-x)} = \frac{3h}{4l}.$$

**87. Position of Equilibrium of a Floating Body.** In general, a body which will float in a liquid will float in more

than one position. In all cases, however, the weight of the liquid displaced is equal to the weight of the body, and therefore the volume of the liquid displaced will be constant.

All planes which cut off this constant volume from the body are possible planes of flotation, and equilibrium will be possible if the line joining the centre of gravity of the body and the centre of buoyancy are vertical. The following theorem in connection with the equilibrium of a floating body is important.

If a plane cut off a volume  $V$  from a body and if the plane be turned through a small angle about a straight line in itself, then the volume  $V$  will remain the same if the straight line passes through the centroid of the area of the plane section.

Let the cutting plane be turned through a small angle  $\theta$  about a line  $Ox$  in the plane.

Then if  $\delta A$  is an element of the plane of rotation, the whole additional volume cut off  $= \int y\theta\delta A$ ; where  $y$  is the distance of that element from  $Ox$ . But this is zero over the whole area,

$$\therefore \theta \int y dA = 0,$$

$$\text{or} \quad \int y dA = 0$$

which is the condition that the centroid of the cutting plane should lie on  $Ox$ .

**88. Stability of Equilibrium of a Floating Body.** A floating body can be given two displacements, a vertical and a rotational one. Any particular displacement might involve both, but we shall suppose that, if the displacement is small, the two changes in position can be treated separately.

For a *small vertical displacement* the forces acting will tend to make the body return to its original position. If, for instance, the body be depressed, an additional upward fluid pressure will be exerted equal to the weight of the excess fluid displaced. In any case, the forces acting tend to restore the body to its original position, therefore the body will always be in stable equilibrium, and will oscillate about the position of equilibrium if slightly displaced.

For a *rotational displacement* we shall assume that the volume of fluid displaced remains unchanged and therefore that the axis of rotation passes through the centroid of the plane of flotation. The centre of gravity of the floating body is also supposed to be in the same vertical plane as the centre of buoyancy.

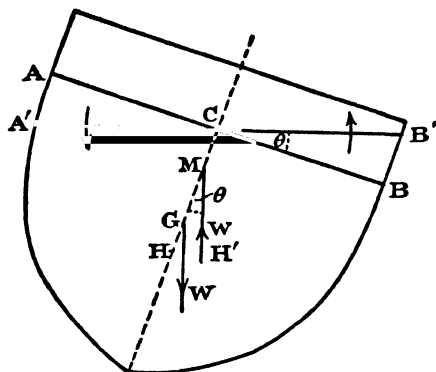


Fig. 140.

Imagine a small angular displacement  $\theta$  to be given to the body about a horizontal axis, symmetrical with regard to the plane of flotation and perpendicular to the plane of the paper.

ACB is the original plane of flotation, A'CB' the new one. G is the centre of gravity of the floating body, W its weight, H is the original centre of buoyancy and H' the new

one, which is assumed to be in the same vertical plane as H and G. Through H' draw a vertical line cutting the original axis of symmetry through HG in M. Then the point M is called the **Metacentre**. If M is above G, the forces acting on the body are an upward force through M and a downward force through G, both equal to W, (since the volume of liquid displaced by the rotation is unaltered), i.e., a couple tending to restore equilibrium. Now moment of restoring couple

$$\begin{aligned} &= W \cdot \overline{MG} \sin \theta \\ &= W \cdot \overline{MG} \theta, \text{ since } \theta \text{ is small.} \quad \dots (1) \end{aligned}$$

This moment may be otherwise expressed, for it is the same as that due to the weight of the wedge of liquid ACA' acting downwards and of BCB' acting upwards together with W acting upwards at H and downwards at G. Since the volumes of the wedges are equal these forces also reduce to a couple.

Let LBNA represent the plane of flotation, and let LN which passes through its centroid be the line about which the surface is rotated.

Let PR and QS be two lines at right angles to LN and take LN as the y axis.

Then area of PRSQ =  $x \delta y$ .

If the body is turned through an angle  $\theta$ , then the weight of the element of the wedge =  $\frac{1}{2} x^2 \theta w \delta y$ , where  $w$  = weight of unit volume of the fluid.

The moment of this force about PQ

$$= \frac{2}{3} x \cdot \frac{1}{2} \delta y \cdot x^2 \theta w.$$

$\therefore$  the moment of the weight of the wedge about LN =  $\frac{w \theta}{3} \int x^3 dy$ .

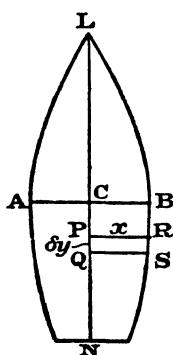


Fig. 141.

But  $\int \frac{1}{2} x^2 dy =$  moment of inertia of the surface LBNC about LN  $= \frac{Ak^2}{2}$ , where  $\frac{A}{2}$  is the area of the surface LBNC and  $k$  is the radius of gyration ;

$\therefore$  moment of restoring couple  $= w\theta \cdot Ak^2 - W \cdot \overline{HG} \theta$ . (2)

Hence, from (1) and (2)

$$w\theta \cdot Ak^2 - W \cdot \overline{HG} \cdot \theta = W \cdot \overline{MG} \cdot \theta$$

$$\therefore wAk^2 = W \cdot \overline{HM}.$$

Now  $W = wV$ , where  $V$  is the volume of the displaced liquid

$$\therefore HM = \frac{Ak^2}{V}.$$

If  $\frac{Ak^2}{V} > HG$ , the equilibrium is stable, since the restoring couple diminishes  $\theta$  ;

if  $\frac{Ak^2}{V} = HG$  the equilibrium is neutral ;

if  $\frac{Ak^2}{V} < HG$  the equilibrium is unstable since the restoring couple increases  $\theta$ .

**Ex. VI.** Find the condition for the stability of equilibrium of a cylinder floating in water with its axis vertical.

Let  $l$  be the length of the cylinder,  $a$  its radius,  $x$  the depth immersed, and  $s$  its specific gravity.

Since weight of cylinder = weight of water displaced,

$$\therefore \pi a^2 l s = \pi a^2 x,$$

$$\therefore x = ls. \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

For stability of equilibrium

$$\frac{Ak^2}{V} > HG ;$$

$$\frac{\pi a^2 \cdot \frac{a^2}{4}}{\pi a^2 x} > \frac{l}{2} - \frac{x}{2}$$

$$\text{i.e.,} \quad a^2 > 2x(l - x),$$

$$\text{i.e.,} \quad a > l\sqrt{2s(1 - s)}$$

Thus if  $s$  is  $\frac{1}{2}$ ,  $l < \frac{3}{2}a$ , so that the length must be less than three-quarters of the diameter.

**Ex. VII.** A cube floats in water with two faces horizontal. Find the conditions under which the equilibrium is stable or unstable.

Let  $a \equiv$  length of edge of cube,  $x \equiv$  length immersed and  $s \equiv$  specific gravity of the solid body.

Now weight of body = weight of water displaced.

$$\therefore a^3 s = a^2 x$$

$$\therefore x = as.$$

$$\text{Again} \quad HM = \frac{Ak^2}{V} = \frac{a^2 \cdot \frac{a^2}{12}}{a^2 x} = \frac{a^2}{12x}$$

$$\text{Also} \quad HG = \frac{1}{2}(a - x).$$

Equilibrium is stable, neutral or unstable

according as  $HM > = \text{or} < HG.$

$$\text{i.e.,} \quad \text{,,} \quad \text{,,} \quad \frac{a^2}{12x} > = \text{or} < \frac{1}{2}(a - x)$$

$$\text{i.e.,} \quad \text{,,} \quad \text{,,} \quad a^2 > = \text{or} < 6x(a - x)$$

$$\text{i.e.,} \quad \text{,,} \quad \text{,,} \quad a^2 > = \text{or} < 6a^2 s(1 - s)$$

$$\text{i.e.,} \quad \text{,,} \quad \text{,,} \quad s^2 - s + \frac{1}{6} > = \text{or} < 0.$$

Hence, for stability,  $s$  cannot lie between

$$\frac{1}{2} + \frac{\sqrt{3}}{6} \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{3}}{6}.$$

Between these values the equilibrium is unstable.

**Ex. VIII.** A cone of specific gravity  $s$ , whose height is  $h$  and the radius of whose base is  $r$ , floats in liquid of specific gravity  $S$  with its axes vertical and vertex upwards. Prove that the equilibrium

is stable if  $\frac{h^2}{r^2 + h^2} < \left(\frac{S - s}{s}\right)^{\frac{1}{3}}.$

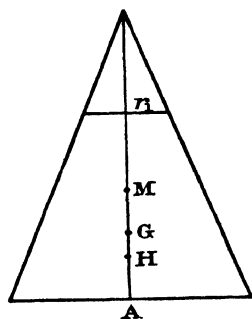


Fig. 142.

Let  $r_1$  be the radius of the base of that portion of the cone above the water, and  $V_1$  its volume.

Let  $x$  be the depth immersed,  $V_2$  the volume of water displaced.

Let  $H$  be the centre of buoyancy,  $G$  the centre of gravity of the cone, and  $A$  the centre of the base.

$$\text{Now } \frac{r_1}{r} = \frac{h - x}{h} \cdot \dots \dots \dots (1)$$

Taking moments about  $A$  to find the length  $AH$ , we have

$$(V_1 + V_2)AG = V_1 \left( x + \frac{h - x}{4} \right) + V_2 \cdot AH;$$

$$\therefore \quad AII = \frac{h}{4} - \frac{3}{4} \frac{V_1}{V_2} x, \quad \text{since } AG = \frac{h}{4},$$

$$\therefore \quad HG = \frac{3}{4} \frac{V_1}{V_2} x.$$

Again  $HM = \frac{Ak^2}{V_2} = \frac{\pi r_1^2 \cdot r_1^2}{4 \cdot V_2}$ .

For stability of equilibrium

$$HG < HM,$$

$$\therefore \frac{3}{4} \frac{V_1 x}{V_2} < \frac{\pi r_1^4}{4 V_2},$$

$$\text{i.e., } x < \frac{r_1^2}{h-x}, \text{ since } V_1 = \frac{1}{3} \pi r_1^2 (h-x)$$

$$\text{i.e., } \frac{x}{h-x} < \frac{r^2}{h^2}, \text{ by (1),}$$

$$\text{i.e., } \frac{h^2}{h^2 + r^2} < \frac{h-x}{h} \dots \dots \dots (2)$$

Now weight of liquid displaced = weight of cone,

$$\therefore \frac{1}{3} \pi \{r^2 h - r_1^2 (h-x)\} S = \frac{1}{3} \pi r^2 h s,$$

$$\text{whence } \frac{S-s}{S} = \left( \frac{h-x}{h} \right)^3, \text{ from (1):}$$

$$\text{therefore } \frac{h^2}{h^2 + r^2} < \left( \frac{S-s}{S} \right)^{\frac{1}{3}}, \text{ from (2).}$$

**89. Experimental Method by which the Position of the Metacentre is Determined.** The position of the metacentre of a ship can be found experimentally by moving a weight across the deck and measuring the angle through which the ship rolls by means of a plumb line.

Let  $W$  be the weight of the ship and  $W'$  the weight of the movable mass.

Let  $\theta$  be the angle of deflection of the ship.

If  $l$  is the length of a plumb line, and  $s$  the arc through which it swings, then

$$\theta = \frac{s}{l}, \theta \text{ being measured in radians.}$$

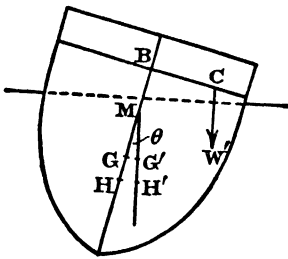


Fig. 143.

Let  $MB = b$

Let  $G$  be the original centre of gravity, and  $H$  the original centre of buoyancy.

Let  $GH$  cut the deck at  $B$ .

Then, if the weight  $W'$  be moved a distance  $c$  across the deck, the centre of gravity will move from  $G$  to  $G'$ ,  $H$  will move to  $H'$ , and  $H'G'$  will cut  $HGB$  in the metacentre  $M$ .

Taking moments about  $G'$  for the equilibrium of the ship, we have

$$W \cdot GM \cdot \sin \theta = W' \cdot (c \cos \theta + b \sin \theta).$$

$$\therefore GM = \frac{W'}{W} \cdot \left( \frac{c}{\theta} + b \right), \text{ as } \theta \text{ is small.}$$

$= \frac{W'c}{W\theta}$  approximately, since  $b$  may for all practical purposes be neglected in comparison with  $\frac{c}{\theta}$ .

$$= \frac{W'cl}{Ws}.$$

Thus in H.M.S. *Achilles* of 9000 tons displacement, 20 tons moved across the deck a distance of 42 feet caused the bob of a pendulum 20 ft. long to move through 10 inches.

$$\therefore GM = \frac{20}{9000} \times \frac{42 \times 20 \times 6}{5} = 2.24 \text{ feet.}$$

**90. Vertical Oscillations of a Floating Body.** A body floating in liquid is slightly depressed. To find the period of oscillation.

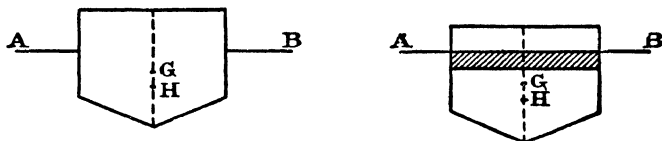


Fig. 144.

Suppose that the surface  $AB$  of the liquid remains at a constant height while the body is depressed a small distance  $x$  below  $AB$ . As  $x$  is small, the additional liquid displaced may be considered to be cylindrical.

Let  $V$  be the volume of the liquid displaced when the body is floating freely, and  $\rho$  its density.

Downward force acting on the body

$$= \text{weight of the body} - \text{the weight of liquid displaced.}$$

$$= -g\rho Ax,$$

where  $A$  is the area of the plane of flotation

Hence, the equation of motion is

$$M \frac{d^2x}{dt^2} = -g\rho Ax, \quad \dots \quad \dots \quad \dots \quad (1)$$

when  $M$  is the mass of the body.

But weight of body = weight of volume  $V$  of the fluid

$$\therefore Mg = g\rho V$$

$$\text{or } M = \rho V.$$

∴ equation (1) becomes

$$\frac{d^2x}{dt^2} = -\frac{gA}{V}x,$$

which is that of Simple Harmonic Motion and of which the solution is

$$x = C \sin \sqrt{\frac{gA}{V}} t;$$

∴ the periodic time is  $2\pi \sqrt{\frac{V}{gA}}$ .

**Ex. IX.** A conical buoy, 4 feet high with a base 3 ft. in diameter, floats with its axis vertical and point downwards in a smooth sea. The buoy weighs 300 lb., and the sea water weighs 64 lb. per cubic foot. If the buoy is slightly depressed, find the time of a vertical oscillation.

Let  $r$  ft. be the radius of the plane of flotation, and  $h$  ft. the depth immersed; then

$$\frac{r}{h} = \frac{1\frac{1}{2}}{4} = \frac{3}{8}. \quad \dots \quad \dots \quad \dots \quad (1)$$

If  $V$  is the volume of the liquid displaced and  $A$  the area of flotation,

$$T = 2\pi \sqrt{\frac{V}{Ag}} = 2\pi \sqrt{\frac{\pi r^2 h}{3\pi r^2 g}} = 2\pi \sqrt{\frac{h}{3g}}. \quad \dots \quad \dots \quad (2)$$

But weight of the body = weight of sea water displaced

$$\begin{aligned} \therefore 300 &= V \times 64 \\ &= \frac{1}{3}\pi r^2 h \times 64 \end{aligned}$$

$$\therefore r^2 h = \frac{900}{64\pi}.$$

$$\text{But } r = \frac{3}{8}h \quad \therefore h = \left(\frac{100}{\pi}\right)^{\frac{1}{3}}$$

from which the periodic time is determined by substituting in (2).

## 91. Equilibrium of a Perfect Gas : Work done in Expansion.

*The Fundamental Equation of a Perfect Gas.*

The characteristic equation of a perfect gas is

$$pv = RT$$

where  $T$  is the absolute temperature of the gas,  $p$  is the intensity of pressure,  $v$  is the volume of unit mass (often called its specific volume) and  $R$  is a constant depending on the nature of the gas.

This equation is derived from the two laws of Boyle and Charles.

The second of these is to the effect that different gases expand between any two given temperatures in the same proportion, if the pressures remain constant. The rate of



expansion per degree Centigrade is  $\cdot 003665$ , or about  $\frac{1}{273}$ , of the volume of the gas at  $0^{\circ}\text{C}$ .

Hence if  $v_0$  denote the volume of the gas at  $0^{\circ}\text{C}$ . and  $v$  its volume at  $t^{\circ}\text{C}$ ., we have

$$v = v_0 \left( 1 + \frac{t}{273} \right).$$

Similarly if  $v'$  denote its volume at  $t'^{\circ}\text{C}$

$$v' = v_0 \left( 1 + \frac{t'}{273} \right).$$

Denoting  $273 + t$  by  $T$ , where  $T$  is the absolute temperature, we have

$$\frac{v}{v'} = \frac{T}{T'}.$$

so that the volume of a gas is proportional to its absolute temperature.

But, by Boyle's Law,  $v$  varies inversely as the intensity of pressure  $p$ , the temperature remaining constant,

$$\therefore v = k \frac{T}{p}, \text{ where } k \text{ is a constant.}$$

If  $v$  is the volume of unit mass, this is written

$$pv = RT. \quad \dots \quad (1)$$

NOTE 2. The value of  $R$  for any gas can be found by substituting any known values of  $p$ ,  $v$  and  $T$ . Thus for atmospheric air at  $0^{\circ}\text{C}$ ., where the height of the mercury barometer is 76 cms., since the density of the air is  $\cdot 00129$  grams per c.c., under those conditions,

$$R = \frac{76 \times 13.59}{\cdot 00129 \times 273} = 2933,$$

approximately, the unit of force being a gramme weight. It is well to note that, in the case of an ordinary gas, equation (1) is only approximately true.

If, during the expansion of a gas, the temperature remain constant, the expansion is said to be *isothermal*; the operation must necessarily be performed so slowly that no heat is absorbed during the process. If on the other hand, a given mass of gas expand, no heat meanwhile being either allowed to escape from it, or enter it from outside, the expansion is said to be *adiabatic*. Such an expansion must evidently take place very rapidly.

When the expansion is adiabatic, the equation connecting the pressure and volume of the gas is

$$pv^{\gamma} = C, \quad \dots \quad (2)$$

where  $\gamma$  is a constant, being in fact the ratio of the specific heat of the gas at constant volume to the specific heat at constant pressure.

The value of  $\gamma$  for air, and several other gases (oxygen, hydrogen, etc.) is approximately 1.41. The reader is referred to works on Thermodynamics for a proof of equation (2).

By combining equation (2) with the fundamental equation  $p v = RT$ , we obtain the adiabatic relations between the pressure and temperature and between the volume and temperature. Thus, if the initial and final states be denoted by  $p_0, v_0, T_0, p_1, v_1, T_1$ , respectively we have

$$p_1 v_1^\gamma = p_0 v_0^\gamma$$

$$\therefore \frac{p_1}{p_0} = \left( \frac{v_0}{v_1} \right)^\gamma \quad \dots \quad \dots \quad \dots \quad (3)$$

and 
$$\frac{T_1}{T_0} = \frac{p_1 v_1}{p_0 v_0} = \left( \frac{v_0}{v_1} \right)^{\gamma-1} = \left( \frac{p_1}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \quad \dots \quad (4)$$

*Work done in the Expansion of a Gas.*

Imagine the gas to be contained within a vessel, capable of expansion, the intensity of pressure on the vessel being  $p$ .

During the small expansion of the gas let each element of area,  $\delta S$ , of the containing vessel be forced outwards.

Let  $\delta n$  be the length of the normal to the surface at any point P contained in the element of area  $\delta S$ .

Then the work done by the pressure on  $\delta S$  is  $p \delta S \delta n$ . But  $\delta S \delta n$  is an element of volume, and if  $v$  is the initial volume of the gas, and  $\delta v$  the expansion

$$\Sigma \delta S \cdot \delta n = \delta v.$$

Again, the intensity of pressure is the same on every element of the containing vessel so that the work done in expanding from  $v$  to  $v + \delta v$

$$= \Sigma p \delta S \delta n = p \Sigma \delta S \delta n = p \delta v.$$

Hence the total work (W) done by the pressure of the gas in expanding from a volume  $v_0$  to a volume  $v_1$  is given by

$$W = \int_{v_0}^{v_1} p dv. \quad \dots \quad \dots \quad \dots \quad (5)$$

If  $p$  is measured in dynes per square centimetre and  $v$  in cubic centimetres, the work is in ergs.

*Work done during an Isothermal Expansion.*

If the temperature during the expansion is constant,

$$p v = C$$

so that (1) becomes

$$W = \int_{v_0}^{v_1} \frac{C}{v} = C \log \frac{v_1}{v_0} = p_0 v_0 \log \frac{v_1}{v_0}. \quad \dots \quad (6)$$

Similarly the work done on a gas during a compression from  $v_0$  to  $v_1$  is

$$W = C \log \frac{v_0}{v_1} = p_0 v_0 \log \frac{v_0}{v_1}. \quad \dots \quad (7)$$

NOTE 3 The amount of work done may be represented in a diagram by taking the volumes of the gas as abscissae and the corresponding intensities of pressure as ordinates, and describing the curve  $pv = C$ . The curve so drawn, which is called an isothermal line will be a rectangular hyperbola, and the work done will be represented by the area enclosed by the curve, the axis of volumes and the ordinates corresponding to the values  $v_0$  and  $v_1$ .

**Ex. X.** An air-tight piston, of weight  $W$ , slides smoothly in a cylinder whose axis is vertical, it is at rest at a height  $h$  from the base of the cylinder, being supported by the compressed air beneath it. An additional weight  $W$  is placed on the piston and the new position of equilibrium is at a depth  $k$  below the former. Find the ratio of the latter pressure of the air in the cylinder to atmospheric pressure.

Let  $\Pi$  be the atmospheric pressure, let  $A$  be the area of cross section of the cylinder,  $p_0, p_1$  the pressures, and  $v_0, v_1$  the volumes.

Since  $p_1 v_1 = p_0 v_0$

$$\therefore \left( \frac{2W}{A} + \Pi \right) A(h - k) = \left( \frac{W}{A} + \Pi \right) A \cdot h.$$

$$\therefore \frac{W}{A}(2h - 2k - h) = \Pi(h - h + k)$$

$$\therefore \frac{W}{A} = \frac{k\Pi}{h - 2k}.$$

$$\begin{aligned} \text{But } p_1 &= \frac{2W}{A} + \Pi \\ &= \Pi \left( \frac{2k}{h - 2k} + 1 \right) \\ &= \frac{h}{h - 2k} \Pi \end{aligned}$$

$$\therefore \text{ratio required} = \frac{h}{h - 2k}.$$

*Work done during an Adiabatic Expansion.*

$$\begin{aligned} \text{Here } W &= \int_{v_0}^{v_1} p dv \\ &= C \int_{v_0}^{v_1} \frac{dv}{v^\gamma}, \quad \text{since } pv^\gamma = C, \\ &= C \left[ -\frac{1}{\gamma-1} \frac{1}{v^{\gamma-1}} \right]_{v_0}^{v_1} \end{aligned}$$

$$= \frac{C}{\gamma - 1} \left[ \frac{1}{v_0^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}} \right] \quad (8)$$

But

$$C = p_0 v_0^\gamma = p_1 v_1^\gamma$$

$\therefore$

$$W = \frac{1}{\gamma - 1} (p_0 v_0 - p_1 v_1). \quad (9)$$

If we consider *one gram* of the substance, then since

$$p_0 v_0 = RT_0 \quad \text{and} \quad p_1 v_1 = RT_1,$$

where  $T_0, T_1$  are the initial and final temperatures, we also have

$$W = \frac{R}{\gamma - 1} (T_0 - T_1), \quad \dots \quad (10)$$

$R$ , the characteristic of the gas, being found as before mentioned.

**Ex. XI.** Find the work done by 3 cubic feet of air at a pressure of 120 lb. per sq. in. expanding to 10 cubic feet assuming that the air obeys (1) Boyle's Law, (2)  $p v^{1.4} = \text{constant}$ .

(1) In the first case

$$\begin{aligned} \text{Work done} &= \int_3^{10} p dv = C \int_3^{10} \frac{dv}{v} \\ &= C \log_e \frac{10}{3}. \end{aligned}$$

But  $C = 120 \times 3 \times 144$ , from the initial conditions.

$$\begin{aligned} \therefore \text{work done} &= \frac{360 \times 144 \times 52287}{.43429} \text{ ft. lb.} \\ &= 62399 \text{ ft. lb.} \end{aligned}$$

(2) In the second case,

$$\begin{aligned} \text{Work done} &= \int_3^{10} p dv = C \int_3^{10} \frac{dv}{v^{1.4}} \\ &= \frac{C}{.4} \left( \frac{1}{3^{.4}} - \frac{1}{10^{.4}} \right) \end{aligned}$$

or, since  $C = 120 \times 144 \times 3^{1.4}$ .

$$\begin{aligned} &= \frac{120 \times 144 \times 3}{.4} \left[ 1 - (.3)^{.4} \right] \text{ ft. lb.} \\ &= 49533 \text{ ft. lb.} \end{aligned}$$

**Ex. XII.** If atmospheric air at  $14^\circ\text{C}$ . is suddenly compressed adiabatically to one tenth of its original volume what are its final temperature and pressure, taking the original barometric height as 76 cm. of mercury, and  $\gamma$  as 1.41?

Here 
$$\frac{T_1}{273 + 14} = (10)^{1.41 - 1},$$
  
 from which  $T_1 = 737.7^\circ,$   
 so that the final temperature is  $464.7^\circ \text{C}.$

Also 
$$\frac{p_1}{13.59 \times 76} = 10^{1.41}$$
  
 from which  $p_1 = 26548 \text{ gm/cm}^2.$

### Problems XII.

1. A solid hemisphere, of radius  $a$ , is totally immersed in water, so that its plane base is vertical and its centre is at a depth  $h$  below the surface of the water. Find the resultant thrust on its curved surface, and show that it is inclined at an angle  $\tan^{-1}\left(\frac{3h}{2a}\right)$  to the vertical.

2. An octant of a sphere is immersed in heavy homogeneous fluid so that one plane face is horizontal; show that the pressures on its three plane faces reduce to a single force, and that the magnitude of this force is  $\frac{1}{15}g\rho a^2(27\pi^2h^2 + 48\pi ha + 32a^2)^{\frac{1}{2}}$ , where  $h$  is the depth of the centre of the sphere and  $a$  its radius.

3. A right circular cone has an angle  $2a$  at the vertex and its specific gravity is  $s$ . Prove that it will float vertically in water with its vertex downwards, in stable equilibrium, if  $\cos^2 a < s^{\frac{1}{3}}$ .

4. A rectangle, of sides  $a$  and  $b$  and of specific gravity  $s$  is floating in water with the side  $b$  vertical. Prove that the equilibrium is stable for a small angular displacement in the plane of the rectangle if  $\frac{a^2}{b^2} > 6s(1 - s).$

5. It is found that, when 10 tons of water are transferred from one side to the other of a vessel of 8000 tons displacement, the bob of a 10 ft. pendulum swings through 6 inches. Show that the height of the metacentre above the centre of gravity is  $13\frac{1}{2}$  inches, given that the breadth of the deck is 45 ft.

6. The stem of a funnel has a radius  $a$  and its mouth has a radius  $b$ ; the height of the stem is  $c$  and that of the slanting part is  $l$ . The mouth is placed on a horizontal plane (being greased so as to remain water-tight) and the funnel is filled with water to the top of the stem; prove that the upward pressure on the funnel is equal to

$$\pi w(b - a)\left\{c(a + b) + \frac{1}{3}l(a + 2b)\right\}.$$

7. A rectangle is hinged along its upper horizontal edge and is partly immersed in water so that when hanging vertically half the rectangle is immersed. The density of the material being  $\cdot 4$ , what is the nature of the equilibrium in the vertical position? If the

rectangle is at rest in another position, what is then its inclination to the vertical?

8. Two thin rods, AB and BC, of the same material and cross-section, are of lengths  $a$  and  $b$  respectively. They are rigidly joined at B and the angle ABC is a right angle. The system hangs from a fixed point by means of a string attached to B and the two rods are partly immersed in water, lengths  $x$  and  $y$  respectively of the rods being out of the water. Prove that the specific gravity of the material of the rods is

$$1 + \frac{x^3 - y^3}{b^2y - a^2x}.$$

9. A thin cylinder is closed at the top and, being initially full of air at atmospheric pressure, is immersed vertically in water till its top is at a depth  $d$  below the surface. Find the height to which the water rises within the cylinder; and show that, if the cylinder be then released, it will sink or rise according as  $d$  is greater or less than  $(r - 1)h - a/r$ , where  $r$  is the ratio of the weight of the water which could fill the cylinder to the weight of the cylinder,  $a$  the height of the cylinder, and  $h$  the height of the water barometer.

10. Prove that if the ratio of the length to the diameter of a uniform solid circular cylinder of specific gravity  $\cdot 5$  lies between  $\frac{1}{2}\sqrt{2}$  and  $1$ , the cylinder cannot float in stable equilibrium in water with its axis either horizontal or vertical.

11. A uniform rod of length  $a$  and specific gravity  $8/9$  is fastened at one end to a point at a distance  $\frac{1}{3}a$  above the surface of a pond, and the other end is immersed in the water. Find the inclination of the rod to the vertical in the position of equilibrium.

12. A thin uniform rod of weight  $w$  and specific gravity  $s$  floats in a vertical position in stable equilibrium when a particle of weight

$W$  is attached to its lower end. Prove that  $W > w\left(\frac{1}{\sqrt{s}} - 1\right)$ .

13. A cylindrical block of wood of length  $L$  cm. and sectional area  $A$  sq. cm floats with its axis vertical in a large expanse of water. If it be pushed down slowly till it is just immersed, prove that the work done is  $\frac{1}{2}gAL^2(1 - s)^2$  ergs where  $s$  is the specific gravity of the wood.

14. A ship with its cargo weighs  $W$ , and the centre of gravity of the displaced water is  $h$  feet below the water line. A weight  $w$  of the cargo is removed, and it is observed that the draught is diminished by  $d$  feet. If the centre of gravity of the displaced water is now  $h'$  feet below the new water line, show that

$$\frac{w}{W} = \frac{d + h' - h}{h' + \frac{d}{2}},$$

the sides of the ship near the water line being assumed vertical.

15. A solid sphere of radius  $r$ , weight  $W$ , and specific gravity  $s$  lies on the bottom of a cylindrical vessel of radius  $a$  and height  $h$ ,

which contains water just up to the top. Prove that the work required to raise the sphere just clear of the water is

$$W \left\{ h - \frac{4r^3}{3a^3} - \frac{1}{s} \left( h - r - \frac{2r^3}{3a^3} \right) \right\}.$$

16. Prove that a cubical block of specific gravity .75 can float in water with one edge in the surface.

17. A sphere of radius  $a$  and specific gravity  $s$  floats in water. When the sphere is given a small vertical displacement it oscillates in the vertical plane of displacement. If the period of these small oscillations is  $2\pi\sqrt{ga/20g}$  find the depth immersed in the equilibrium position, and the value of  $s$ .

18. A cylinder, height  $h$ , radius  $a$  and specific gravity  $s$  floats with its axis vertical in water. If it be given a small vertical displacement, find the periodic time of the resulting vibration.

19. A quantity of gas expands so as to satisfy the law  $p v = C$ , a constant. Find the work done in expansion from  $v = 2$  cub. ft. to  $v = 10$  cub. ft., having given that at atmospheric pressure (15 lb. wt. per sq. in.) the volume is 20 cub. ft.

20. A cubic foot of air at 100 lb. per sq. in. pressure expands down to atmospheric pressure (14.7 lb. per sq. in.) following Boyle's law. Show that the work done is 27600 ft. lb. approximately.

21. A quantity of steam expands so as to satisfy the law  $p v^{1.18} = C$ , a constant. Find the work done in expansion from  $v = 3$  cub. ft. to  $v = 15$  cub. ft., have given that when the volume is 12 cub. ft. the pressure is 4000 lb. wt. per sq. ft.

22. If a quantity of air expands in such a manner that no heat is lost or gained then  $p v^c$  remains constant, where  $c = 1.41$ . Prove that the work done when the initial pressure and volume are  $p_1$  and  $v_1$  and the final pressure and volume are  $p_2$  and  $v_2$ , is

$$\frac{1}{c-1} (p_1 v_1 - p_2 v_2).$$

Find  $p_2$  and the work done if  $v_1 = 10$  c.in.,  $v_2 = 20$  c.in. and  $p_1 = 15$  lb. per sq. in.

23. A given mass of gas is compressed adiabatically from volume  $v$  to volume  $v'$ . Find the work done.

By what fraction of its length should a column of atmospheric air be compressed in order that its temperature may increase from  $10^\circ\text{C}$ . to  $200^\circ\text{C}$ .? [Ratio of spec. heats of air at constant pressure and at constant volume may be taken as 1.4.]

24. Calculate the height to which the water will rise in a cylindrical diving bell, 12 ft. high, when its top is lowered to a depth of 60 feet, being given that the temperature of air at the surface is  $80^\circ\text{F}$ ., height of water barometer at surface 33 ft., and temperature of water  $40^\circ\text{F}$ .

25. A circular cylinder, of cross section  $A$  and height  $a$ , is full of air at atmospheric pressure  $H$ ; the air is slowly compressed by means of an air-tight piston until it fills a height  $b$  of the cylinder;

the temperature being supposed to remain constant, show that the work done during the compression is

$$A\Pi \left[ -a + b \right].$$

26. A pump has a stroke of 12 inches and is used to force air into a motor tyre in which the pressure is 60 lbs. per sq. in. by gauge. What length of stroke of the pump will be swept through before air begins to enter the tyre, (i.) if  $p\nu$  remains constant and (ii.) if  $p\nu^{1.4}$  remains constant during the compression. The barometric pressure is 15 lbs. per sq. in.

27. A safety device for lifts consists of an extension of the lift shaft below ground level; the floor of the lift is made to fit this well closely, so that a pneumatic buffer is provided.

A lift weighing 3000 lbs. falls from a height of 30 ft. above ground level into such a safety pit 10 ft. deep, the base of the lift being 8 ft. by 5 ft. Find approximately how far the lift will descend before it is stopped, neglecting air leakage and assuming that the pressure of the air varies inversely as its volume. The resulting equation may be solved graphically: take atmospheric pressure as 15 lbs. per sq. in.

28. A hollow vessel in the shape of a paraboloid of revolution floats in water with its axis vertical, and vertex downwards. If the weight of the vessel itself may be neglected, find approximately to what height it must be filled with mercury (sp. gr. 13.6) in order that its vertex may be 18 inches below the free surface of the water.

29. A sphere of radius 6 cm. and sp. gr. .5 floats freely in a cylindrical jar of water of very slightly larger radius. How much must the sphere be raised so that it may just clear the surface of the water? How much work will have to be done to effect this?

30. Assuming that the rate at which gas is discharged from a cylinder through a nozzle is proportional to the excess of pressure above atmospheric pressure and is 20 c.cm. per sec. (measured at atmospheric pressure) when the pressure in the cylinder is 11 atmospheres, find the time taken to half empty a cylinder of 8 litres capacity, the pressure being originally 20 atmospheres. Find also the pressure in the cylinder at the end of 40 minutes from the opening of the tap.



## CHAPTER XIII

### FURTHER EXAMPLES ON LINEAR MOTION

#### 92. Fall of a Heavy Particle in a Resisting Medium.

When a body is moving in a fluid, the friction between the body and the fluid causes a retardation, which follows no definite law. For very small velocities, the frictional resistance will be approximately proportional to the velocity, for greater velocities it will approximate to the square of the velocity, and for very high velocities the cube of the velocity.

It should be noticed that we are not dealing with a conservative system of forces.

(i.) *Case of a body falling under gravity, the resistance of the air being assumed proportional to the velocity.*

The equation of motion will then be

$$\frac{dv}{dt} = g - kv, \quad \dots \dots \dots (1)$$

$k$  being a constant, and the velocity being measured downwards;

$$\int \frac{dv}{g - kv} = \int dt;$$

$$-\frac{1}{k} \log \left( \frac{g - kv}{C} \right) = t$$

where  $C$  is a constant.

$$\therefore g - kv = Ce^{-kt} \quad \dots \dots \dots (2)$$

This gives the velocity at any instant.

As  $t$  increases  $e^{-kt}$  decreases, and therefore if  $t$  increases indefinitely

$$g - kv = 0,$$

i.e., 
$$v = \frac{g}{k}.$$

This value to which the velocity continually approaches is called the terminal velocity.

\*Equation (1) could have been written

$$\ddot{x} = g - k\dot{x} \quad \dots \dots \dots (3)$$

and, integrating once, we obtain

$$\dot{x} + kx = gt \quad \dots \dots \dots (4)$$

if the velocity is zero when  $x = 0$  and  $t = 0$ .

\* In this and following chapters the dotted notation for differential coefficients with respect to  $t$  (Art. 21) will be used for the sake of brevity.

Equation (4) can be integrated if both sides are multiplied by  $e^{kt}$ , for

$e^{kt} \dot{x} + ke^{kt}x$  is the differential coefficient of  $xe^{kt}$ .

Hence equation (4) may be written

$$\frac{d}{dt}(xe^{kt}) = e^{kt}gt.$$

Integrating,

$$\begin{aligned} xe^{kt} &= \int e^{kt}gt \, dt + N \\ &= \frac{ge^{kt}}{k^2}(kt - 1) + N. \quad \dots \quad (5) \end{aligned}$$

Putting  $x = 0$ , when  $t = 0$ ,  $N = \frac{g}{k^2}$ , and equation (5) becomes

$$x = \frac{gt}{k} - \frac{g}{k^2} + \frac{g}{k^2}e^{-kt} \quad \dots \quad (6)$$

which gives the space passed over in time  $t$ .

By writing  $v = \frac{dx}{dt}$  in (2), we can obtain the same result as in (6), remembering that  $C = g$ , since we have assumed that the particle started from rest.

(2) *Case of a body falling under gravity, the resistance of the air being assumed proportional to the square of the velocity.*

The equation of motion is

$$\ddot{x} = g - k\dot{x}^2 \quad \dots \quad (1)$$

$$\text{i.e.,} \quad \frac{dv}{dt} = g - kv^2 \quad \dots \quad (2)$$

$$\therefore \quad \frac{1}{k} \int \frac{dv}{\frac{g}{k} - v^2} = t + C.$$

Integrating, we have

$$\frac{1}{2k} \log \frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v} = t + C. \quad \dots \quad (3)$$

Writing  $V^2 = \frac{g}{k}$  this becomes

$$\log \frac{V + v}{V - v} = 2kV(t + C). \quad \dots \quad (4)$$

If the body start from rest, then  $C = 0$

and 
$$\log \frac{V+v}{V-v} = 2kVt, = \frac{2g}{V}t,$$

$\therefore t = \frac{V}{2g} \log \frac{V+v}{V-v}, \quad \dots \dots (5)$

which gives the time required for the particle to attain to a velocity  $v$ .

Now 
$$\frac{V+v}{V-v} = e^{2kVt};$$

$\therefore \frac{v}{V} = \frac{e^{2kVt} - 1}{e^{2kVt} + 1} \quad \dots (6)$

$$= \frac{e^{kVt} - e^{-kVt}}{e^{kVt} + e^{-kVt}} \\ = \tanh kVt.$$

i.e., 
$$v = V \tanh \frac{gt}{V}. \quad \dots (7)$$

The last equation gives the velocity  $v$  after time  $t$ .

From equation (7)

$$\frac{dx}{dt} = V \tanh \frac{gt}{V}.$$

$\therefore x = \frac{V^2}{g} \log \cosh \frac{gt}{V} \quad \dots \dots (8)$

$$= \frac{V^2}{g} \log \frac{e^{\frac{gt}{V}} + e^{-\frac{gt}{V}}}{2}, \quad \dots (9)$$

no additive constant being necessary since  $x = 0$  when  $t = 0$ .

From equation (6) we see that as  $t$  increases indefinitely  $v$  approaches the value  $V$ .

Hence  $V$  is the terminal velocity, and is equal to  $\sqrt{\frac{g}{k}}$ .

The velocity  $v$  may be obtained in terms of  $x$  by eliminating  $t$  between equations (5) and (9).

From (9),

$$e^{kx} = \frac{e^{kVt} + e^{-kVt}}{2}.$$

Squaring, 
$$4e^{2kx} = e^{2kVt} + e^{-2kVt} + 2$$

$$= \frac{V+v}{V-v} + \frac{V-v}{V+v} + 2, \quad \text{from (5)}$$

$$= \frac{4V^2}{V^2 - v^2}.$$

$$\begin{aligned} \therefore v^2 &= V^2 (1 - e^{-2kx}) \\ &= V^2 \left( 1 - e^{-\frac{2g}{V^2}x} \right) \quad \dots \quad (10) \end{aligned}$$

Equation (10) might have been obtained from the original equation of motion ; for

$$\ddot{x} = v \frac{dv}{dx} = \frac{1}{2} \frac{d(v^2)}{dx};$$

and therefore equation (1) becomes

$$\frac{d(v^2)}{dx} + 2kv^2 = 2g \quad \dots \quad (11)$$

This is of the same form as equation (4) of the last article, the integrating factor in this case being

$$e^{\int 2k dx}$$

$$\begin{aligned} \text{Therefore} \quad v^2 e^{2kx} &= 2g \int e^{2kx} dx + B \\ &= \frac{g}{k} e^{2kx} + B. \end{aligned}$$

If the body start from rest, and  $x$  is measured from this point, then  $B = -\frac{g}{k}$ ;

$$\begin{aligned} \therefore v^2 &= \frac{g}{k} (1 - e^{-2kx}) \\ &= V^2 \left( 1 - e^{-\frac{2g}{V^2}x} \right) \quad \dots \quad (12) \end{aligned}$$

the same result as in equation (10).

(3) *Case of a body projected upwards with initial velocity  $u$  against the same resistance as before.*

Measuring  $x$  upwards, the equation of motion will be

$$\ddot{x} = -g - k\dot{x}^2, \quad \dots \quad (1)$$

$$\text{i.e.,} \quad v \frac{dv}{dx} = -g - kv^2; \quad \dots \quad (2)$$

$$\therefore \frac{1}{2k} \int \frac{d(v^2)}{\frac{g}{k} + v^2} = - \int dx.$$

Integrating,  $\log \left( \frac{\frac{g}{k} + v^2}{C} \right) = -2kx$

i.e.,  $\frac{g}{k} + v^2 = Ce^{-2kx}$ . .. .. (3)

When  $x = 0, v = u \quad \therefore C = \frac{g}{k} + u^2$

and, writing  $\frac{g}{k} = V^2$ , we have

$$\frac{V^2 + v^2}{V^2 + u^2} = e^{-\frac{2gx}{V^2}} \quad \dots \quad (4)$$

$$\therefore v^2 = (V^2 + u^2)e^{-\frac{2gx}{V^2}} - V^2. \quad \dots \quad (5)$$

The particle will eventually come to rest, the height,  $h$ , to which it rises being found by putting  $v = 0$  in equation (4).

We then have

$$\frac{V^2 + u^2}{V^2} = e^{\frac{2gh}{V^2}}.$$

$$\therefore h = \frac{V^2}{2g} \log \left( 1 + \frac{u^2}{V^2} \right). \quad \dots \quad (6)$$

The particle then begins to descend, and taking the height reached as the origin, and using the results of the previous article, we can completely determine the motion.

It is still necessary to find the velocity and distance described in terms of the time.

From (1)

$$\frac{dv}{dt} = -g - kv^2,$$

$$\therefore \frac{1}{k} \int \frac{dv}{V^2 + v^2} = A - t$$

$$\therefore \frac{1}{V} \tan^{-1} \frac{v}{V} = \frac{g}{V^2} (A - t),$$

$$\therefore \tan^{-1} \frac{v}{V} = \frac{g}{V} (A - t). \quad \dots \quad (7)$$

When  $t = 0, v = u$ ,

$$\therefore \tan^{-1} \frac{u}{V} = \frac{g}{V} A. \quad \dots \quad (8)$$

$$\begin{aligned}
 \text{From (7)} \quad \frac{v}{V} &= \tan \frac{g}{V} (A - t) \\
 &= \frac{\tan \frac{gA}{V} - \tan \frac{gt}{V}}{1 + \tan \frac{gA}{V} \tan \frac{gt}{V}} \\
 &= \frac{u - \tan \frac{gt}{V}}{1 + \frac{u}{V} \cdot \tan \frac{gt}{V}} \quad \text{from (7)} \\
 &= \frac{u - V \tan \frac{gt}{V}}{V + u \tan \frac{gt}{V}}, \quad \dots \dots (9)
 \end{aligned}$$

which gives the velocity in terms of  $t$ .

$$\text{From (9)} \quad \frac{1}{V} \frac{dx}{dt} = \frac{u \cos \frac{gt}{V} - V \sin \frac{gt}{V}}{V \cos \frac{gt}{V} + u \sin \frac{gt}{V}}. \quad \dots (10)$$

The numerator of this fraction is  $\frac{V}{g}$  times the differential co-efficient of the denominator.

$$\begin{aligned}
 \text{Integrating,} \quad \frac{1}{V} x &= \frac{V}{g} \log \frac{V \cos \frac{gt}{V} + u \sin \frac{gt}{V}}{C} \\
 \text{i.e.,} \quad Ce^{\frac{gx}{V^2}} &= V \cos \frac{gt}{V} + u \sin \frac{gt}{V}. \\
 \text{Now} \quad x = 0 \quad \text{when } t = 0 \quad \therefore C &= V \\
 \therefore Ve^{\frac{gx}{V^2}} &= V \cos \frac{gt}{V} + u \sin \frac{gt}{V} \dots \dots (11)
 \end{aligned}$$

**Ex. I.** Find the loss in kinetic energy, in the last case, when the body has fallen back to the point of projection after being projected upwards with velocity  $u$ .

The particle reaches the height  $h$  given by

$$h = \frac{V^2}{2g} \log \left( 1 + \frac{u^2}{V^2} \right).$$

The velocity at any point during the descent is given by

$$v^2 = V^2 \left( 1 - e^{-\frac{2gx}{V^2}} \right).$$

The velocity of the body when it reaches the point of projection is found by substituting  $h$  for  $x$ ,

$$\text{and therefore} \quad v^2 = V^2 \left\{ 1 - \frac{V^2}{V^2 + u^2} \right\}.$$

$$= \frac{u^2 V^2}{V^2 + u^2}.$$

$$\begin{aligned} \text{Loss of kinetic energy} &= \frac{1}{2} m u^2 - \frac{1}{2} m v^2 \\ &= \frac{1}{2} m u^2 \left\{ 1 - \frac{V^2}{V^2 + u^2} \right\} \\ &= \frac{1}{2} m u^2 \cdot \left( \frac{u^2}{V^2 + u^2} \right). \end{aligned}$$

**Ex. II.** In the case of a body falling from rest in a medium, the resistance of which is proportional to the square of the velocity, prove that when  $x$  is very large, the space traversed in the same time by a point moving with uniform velocity  $\sqrt{\frac{g}{k}}$  exceeds  $x$  by  $\frac{1}{2k} \log_e 2$  approximately.

We have from equation (12) of case (2) that

$$\begin{aligned} v &= \sqrt{\frac{g}{k}} \left( 1 - e^{-2kx} \right)^{\frac{1}{2}} \\ &= \sqrt{\frac{g}{k}} \left( 1 - \frac{1}{2e^{2kx}} \right) \quad \text{approximately.} \end{aligned}$$

$$\begin{aligned} &= \sqrt{\frac{g}{k}} \frac{2e^{2kx} - 1}{2e^{2kx}} \\ \frac{dt}{dx} &= \sqrt{\frac{k}{g}} \frac{2e^{2kx}}{2e^{2kx} - 1} \end{aligned}$$

$$\begin{aligned} \text{i.e.,} \quad \sqrt{\frac{g}{k}} t &= \int_0^x \frac{2e^{2kx}}{2e^{2kx} - 1} \cdot dx \\ &= \frac{1}{2k} \log \left( 2e^{2kx} - 1 \right) \\ &= \frac{1}{2k} \log 2e^{2kx} \left( 1 - \frac{1}{2e^{2kx}} \right) \\ &= \frac{1}{2k} \left( \log_e 2 + 2kx \right) \quad \text{approximately} \end{aligned}$$

$$\therefore \quad x = \sqrt{\frac{g}{k}} t - \frac{1}{2k} \log_e 2 \quad \text{approximately.}$$

But  $\sqrt{\frac{g}{k}} \cdot t$  is the space described with uniform velocity  $\sqrt{\frac{g}{k}}$  in the same time.

∴ space traversed differs from that of a point moving with uniform velocity  $\sqrt{\frac{g}{k}}$  during the same interval of time by  $\frac{1}{2k} \log_e 2$ .

**93. Investigation of the Motion of a Body given by an Equation of the form  $\ddot{x} + ax + bx = f(t)$ , where  $a$  and  $b$  are constants.**

*I. Free Oscillations.*—A particle suspended at the end of an elastic string.

If  $x$  is the extension at any time  $t$ , then by Hooke's Law the tension is  $kx$ , where  $k$  is a constant.

The equation of motion is

$$m\ddot{x} = mg - kx$$

where  $m$  is the mass of the particle,

$$\text{i.e.} \quad \ddot{x} + \frac{kx}{m} = g. \quad \dots \dots (1)$$

The equation may be written

$$\ddot{x} + \frac{k}{m} \left( x - \frac{mg}{k} \right) = 0$$

and writing  $y = x - \frac{mg}{k}$ , the equation becomes

$$\ddot{y} + \frac{k}{m} y = 0, \quad \dots \dots (2)$$

which is the equation for simple harmonic motion.

The solution of (2) is

$$y = A \sin \left( \sqrt{\frac{k}{m}} \cdot t + \alpha \right),$$

$$\text{from which} \quad x = \frac{mg}{k} + A \sin \left( \sqrt{\frac{k}{m}} \cdot t + \alpha \right) \quad \dots (3)$$

where  $A$  and  $\alpha$  are to be determined from the initial conditions.

Equation (3) shows that the particle executes simple harmonic oscillations about the point where the extension of the string is  $\frac{mg}{k}$ , that is, about the point where the tension in the string is equal to the weight of the particle.

*II. Forced Oscillations.*

Suppose now that when the particle is displaced  $x$  feet downwards, the point of suspension of the string is also  $y$  feet below its old position. The elastic string is therefore only elongated by the amount  $x - y$ , and the tension of the string is now  $k(x - y)$ .

The equation of motion is

$$m\ddot{x} = mg - k(x - y). \quad \dots \dots (4)$$



If the force which moves the point of support is a periodic one, this force will give rise to what is called a forced vibration of the particle.

Let  $y = a \sin pt$  be the motion given to the point of support, then (4) becomes

$$m\ddot{x} = mg - k(x - a \sin pt)$$

i.e., 
$$\ddot{x} + \frac{k}{m}\left(x - \frac{mg}{k}\right) = \frac{ak}{m} \sin pt. \quad \dots (5)$$

Writing  $z = x - \frac{mg}{k}$  and  $n^2 = \frac{k}{m}$ , the equation becomes

$$\ddot{z} + n^2 z = n^2 a \sin pt. \quad \dots (6)$$

The complete solution of this form of equation consists of two parts :

- (i.) the Particular Integral which is any solution of (6) ;
- (ii.) the solution obtained when the right hand side is made zero.

Their sum, which must contain two arbitrary constants, is the General Solution.

For the particular integral assume that

$$z = A \sin pt,$$

since the second differential co-efficient of a sine is also a sine.

Differentiating twice, substituting in (6) and equating co-efficients we obtain

$$A = \frac{n^2 a}{n^2 - p^2}$$

and the particular integral is therefore

$$z = \frac{n^2 a}{n^2 - p^2} \cdot \sin pt. \quad \dots (7)$$

The corresponding equation with the right hand side zero is

$$\ddot{z} + n^2 z = 0$$

from which  $z = B \sin (nt + \alpha).$

The complete solution is thus found to be

$$z = B \sin (nt + \alpha) + \frac{n^2 a}{n^2 - p^2} \cdot \sin pt$$

$$\text{or } x = \frac{mg}{k} + B \sin (nt + \alpha) + \frac{n^2 a}{n^2 - p^2} \sin pt. \quad \dots (8)$$

The last equation shows that the particle has two displacements superposed, one due to the free oscillation and the other due to the motion of the point of support, the two displacements being independent of one another. It is for this reason we say that there is a forced vibration of the particle. The period of the free oscillation is

$$\frac{2\pi}{n} \quad \text{or} \quad 2\pi \sqrt{\frac{m}{k}}, \quad \text{while the period of the forced oscillation is } \frac{2\pi}{p}.$$

If  $p$  is nearly equal to  $n$ , that is, if the periods of the two oscillations are nearly equal, the amplitude of the forced vibration, which is

$$\frac{a}{1 - \frac{p^2}{n^2}},$$

will become very great, so that the free oscillation of the particle is negligible.

If  $p$  is very great compared with  $n$ , the forced vibration will be negligible, and the particle will oscillate as if the point of support were at rest.

For the exceptional case when  $p$  is equal to  $n$ , we cannot assume  $x = A \sin nt$  as a particular integral, for no value of  $A$  can then be obtained. It will be found that a solution is  $x = -\frac{nat}{2} \cos nt$ , so that the complete integral is

$$x = \frac{mg}{k} + B \sin (nt + a) - \frac{nat}{2} \cos nt. \quad \dots \quad (9)$$

**III. Damped Oscillations.** Consider next the case of a particle making small oscillations in a gas or fluid, the resistance of which is assumed to be proportional to the velocity.

The equation of motion will be

$$\ddot{x} = -n^2x - k\dot{x}$$

where  $k$  is a constant,

$$\text{i.e.,} \quad \ddot{x} + k\dot{x} + n^2x = 0. \quad \dots \quad (10)$$

$$\text{Let} \quad x = Ae^{\lambda t};$$

$$\text{then} \quad \ddot{x} + k\dot{x} + n^2x = Ae^{\lambda t} (\lambda^2 + k\lambda + n^2)$$

If  $\lambda$  is chosen so that

$$\lambda^2 + k\lambda + n^2 = 0$$

$$\text{then} \quad x = Ae^{\lambda t}$$

will be a solution of the equation.

$$\text{Now} \quad \lambda = \frac{-k \pm \sqrt{k^2 - 4n^2}}{2}$$

and three cases arise according as  $k^2$  is greater than, equal to, or less than  $4n^2$ .

$$(a) \quad k^2 > 4n^2.$$

The two values of  $\lambda$  are now both real and both negative. If we denote them by  $-\alpha$  and  $-\beta$ , the solution of (10) is

$$x = Ae^{-\alpha t} + Be^{-\beta t} \quad \dots \quad (11)$$

two independent constants being necessary.

As  $t$  increases the value of  $x$  continually diminishes. There are no oscillations, but the particle gradually approaches the

position of equilibrium, which it finally reaches in an infinite time.

(b)  $k^2 < 4n^2$ .

The two values of  $\lambda$  are now

$$-\frac{k}{2} + ib, \quad -\frac{k}{2} - ib,$$

where

$$4n^2 - k^2 = 4b^2,$$

so that  $x = Ae^{\left(-\frac{k}{2} + ib\right)t} + Be^{\left(-\frac{k}{2} - ib\right)t}$

$$= e^{-\frac{k}{2}t} (Ae^{ibt} + Be^{-ibt})$$

$$= e^{-\frac{k}{2}t} \{A(\cos bt + i \sin bt) + B(\cos bt - i \sin bt)\}$$

$$= e^{-\frac{k}{2}t} (L \cos bt + M \sin bt)$$

where  $A + B = L$  and  $i(A - B) = M$

i.e.,  $x = Ce^{-\frac{k}{2}t} \sin(bt + \alpha) \dots \dots (12)$

We can regard this as simple harmonic motion with a constant period  $\frac{2\pi}{b}$ , i.e.  $\frac{4\pi}{\sqrt{4n^2 - k^2}}$ , but decreasing amplitude, since  $e^{-\frac{k}{2}t}$  decreases as  $t$  increases and finally vanishes when  $t$  becomes infinitely large.

Oscillations which progressively diminish in amplitude are said to be damped.

Now the period of the undamped oscillation is  $\frac{2\pi}{n}$  so that friction has increased the period. If  $k$  is small the alteration in the period is small, but the amplitude may diminish at a sensible rate.

The curves drawn are those whose equations are

$$x = 3 \sin 3t$$

and

$$x = 3e^{-\frac{1}{2}t} \sin 3t.$$

(c) In the special case when  $k^2 = 4n^2$ , the two values of  $\lambda$  are both real and equal.

To solve the differential equation (10) in these circumstances we return to equation (11) writing  $\beta = \alpha - h$ , where  $h$  is ultimately to be taken indefinitely small.

We then have

$$x = Ae^{-\alpha t} + Be^{-(\alpha - h)t}$$

$$\begin{aligned}
 &= Ae^{-at} + Be^{-at} \left\{ 1 + ht + \text{terms containing higher powers of } h \right\} \\
 &= (A + B)e^{-at} + Bht \cdot e^{-at}, \text{ neglecting higher powers of } h.
 \end{aligned}$$

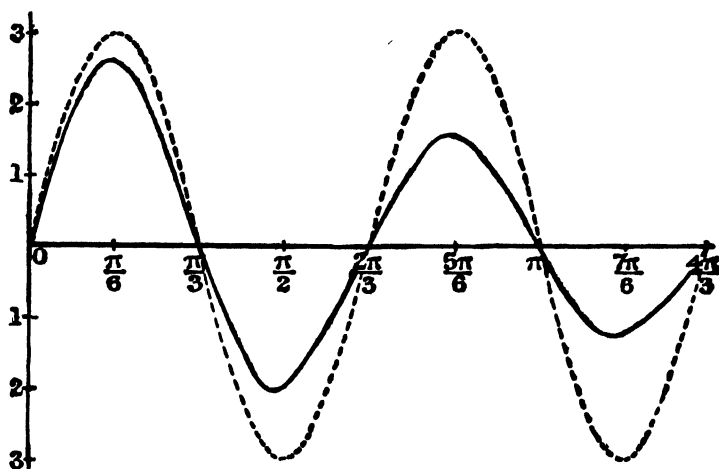


Fig. 145.

Now  $A$  and  $B$  are two independent constants, and we can choose them so that when  $h$  becomes indefinitely small,  $Bh$  remains finite and equal to  $B_1$ , while  $A$  and  $B$  are to be taken of opposite sign, so that their difference also remains finite and equal to  $A_1$ .

We may therefore write

$$x = (A_1 + B_1 t)e^{-at} = (A_1 + B_1 t)e^{-\frac{h}{2}t} \quad \dots \quad (13)$$

as the required solution.

NOTE 1. The student will observe that the solution of a differential equation of the second order must contain two independent constants. In the case where the two roots of  $\lambda$  are equal, equation (11) becomes

$$x = (A + B)t e^{-at}$$

or

$$x = L e^{-at}$$

so that there is only one constant. Equation (13) is therefore an appropriate

solution, and since  $te^{-\frac{h}{2}t}$  approaches zero as  $t$  approaches infinity, the equation represents motion in which the particle gradually approaches its equilibrium position. If  $A$  and  $B$  are of opposite sign, the particle may pass through its equilibrium position once before coming to rest, but the motion is not oscillatory.

NOTE 2. In the motion represented by equation (12), where the periodic time  $T$  is  $\frac{2\pi}{b}$ , let  $x_1, x_2, x_3$ , etc. denote the successive amplitudes after times

$\frac{T}{2}$ ,  $T$ ,  $\frac{3T}{2}$ , etc. Then, neglecting the signs of  $x_1$ ,  $x_2$ ,  $x_3$ , etc., which are alternately positive and negative, we have

$$\frac{x_2}{x_1} = \frac{C e^{-\frac{k}{2} \cdot T}}{C e^{-\frac{k}{2} \cdot \frac{T}{2}}} = e^{-k \frac{T}{4}}.$$

Similarly 
$$\frac{x_3}{x_2} = \frac{x_4}{x_3} = \dots = e^{-k \frac{T}{4}}.$$

so that the amplitudes form a decreasing geometrical progression of ratio  $e^{-k \frac{T}{4}}$ .  
Taking logs we have

$$\log_e x_1 - \log_e x_2 = k \frac{T}{4},$$

$$\log_e x_2 - \log_e x_3 = k \frac{T}{4},$$

$$\log_e x_{n-1} - \log_e x_n = k \frac{T}{4},$$

or, on addition,

$$\frac{\log_e x_1 - \log_e x_n}{n-1} = k \frac{T}{4}.$$

The value  $k \frac{T}{4}$  is called the logarithmic decrement of the damped oscillations

*IV. Forced Oscillations in a medium the resistance of which is proportional to the velocity.*

The equation of motion will be of the form

$$\ddot{x} + n^2 x + k\dot{x} = a \sin pt \quad \dots \quad (14)$$

where  $a \sin pt$  is the periodic force on unit mass, and the integral will consist of two parts.

Assuming as the particular integral

$$x = A \sin pt + B \cos pt,$$

differentiating, substituting in equation (14) and equating corresponding terms, we find

$$A = \frac{(n^2 - p^2)a}{(n^2 - p^2)^2 + p^2 k^2}, \quad B = -\frac{pka}{(n^2 - p^2)^2 + p^2 k^2}.$$

The integral may also be written in the form

$$x = L \sin (pt + \alpha) \quad \dots \quad (15)$$

where

$$L = \frac{a}{\sqrt{(n^2 - p^2)^2 + p^2 k^2}},$$

$$\cos \alpha = \frac{n^2 - p^2}{\sqrt{(n^2 - p^2)^2 + p^2 k^2}}$$

and

$$\sin \alpha = -\frac{pk}{\sqrt{(n^2 - p^2)^2 + p^2 k^2}}.$$

The second part of the complete integral is to be found by

equating the left hand side of equation (14) to zero and has already been determined in case 3.

The value  $L$ , found above, is the amplitude of the forced oscillations. Owing to the resistance of the medium, a term in  $k$  occurs in the denominator of  $L$ , and the form of this denominator shows that the amplitude in these circumstances is always finite.

NOTE 3. The above investigation of the motion of a body given by an equation of the general form

$$\ddot{x} + a\dot{x} + bx = f(t)$$

applies to any small oscillation of a particle about a position of stable equilibrium.

Thus if a particle is constrained to move along a given curve, as, for example, the centre of mass of the bob of a pendulum, the dependent variable will be  $s$ , the space described along the curve from the position of equilibrium, or the corresponding angular displacement  $\theta$ .

**Ex. III.** A particle is attached to a fixed point in a rough horizontal plane by means of an elastic string; the string is drawn out horizontally to its natural length, and the particle is projected along the plane away from the fixed point with velocity  $u$ . Find where the particle first comes to rest and show that it will remain permanently at rest or return towards the fixed point according as  $u$  is less or greater than  $\mu(3gb)^{\frac{1}{2}}$ , where  $\mu$  is the coefficient of friction, and  $b$  is the extension which would be produced by applying a statical force equal to the weight of the particle at the end of the elastic string.

Measuring  $x$  from the point of projection, the equation of motion for motion away from the fixed point is

$$\ddot{x} = -\mu g - kx \quad \dots \quad \dots \quad \dots \quad (1)$$

since the tension in the string is proportional to the extension. Multiplying (1) all through by  $2\dot{x}$  and integrating, we have

$$\dot{x}^2 = A - 2\mu g x - kx^2. \quad \dots \quad \dots \quad \dots \quad (2)$$

But  $\dot{x} = u$ , when  $x = 0 \therefore A = u^2$ .

Hence, if  $v$  is the velocity of the particle,

$$v^2 = u^2 - 2\mu g x - kx^2. \quad \dots \quad \dots \quad \dots \quad (3)$$

The velocity will be zero when

$$kx^2 + 2\mu g x - u^2 = 0$$

$$\text{or when} \quad x = -\frac{\mu g}{k} \pm \frac{\sqrt{\mu^2 g^2 + u^2 k}}{k}. \quad \dots \quad \dots \quad \dots \quad (4)$$

The particle will first come to rest on the positive side of the point of projection, where

$$x = -\frac{\mu g}{k} + \frac{\sqrt{\mu^2 g^2 + u^2 k}}{k} \quad \dots \quad \dots \quad \dots \quad (5)$$

and this is the maximum distance of the particle from the point of projection in the direction of projection.

The tension in the string is then

$$mk \left( -\frac{\mu g}{k} + \frac{\sqrt{\mu^2 g^2 + u^2 k}}{k} \right),$$

and this is also a maximum.



Then, the equation of motion is

$$\begin{aligned}\ddot{x} &= -n^2x + \lambda^2(a - x) \\ \ddot{x} &= -(n^2 + \lambda^2)x + \lambda^2a \quad \dots \quad \dots \quad \dots \quad (5)\end{aligned}$$

$$\text{or, as above, } \frac{d}{dx}(v^2) = -2(n^2 + \lambda^2)x + 2\lambda^2a. \quad \dots \quad \dots \quad (6)$$

$$\text{Integrating, } v^2 = -(n^2 + \lambda^2)x^2 + 2\lambda^2ax + D. \quad \dots \quad (7)$$

$$\text{When } x = a, \quad v^2 = V^2 = 8a^2n^2, \quad \text{from (4).}$$

$$\therefore 8a^2n^2 = -(n^2 + \lambda^2)a^2 + 2\lambda^2a^2 + D,$$

$$\text{whence } D = a^2(9n^2 - \lambda^2).$$

$$\text{But } v = 0 \quad \text{when } x = 0 \quad \therefore D = 0$$

$$\therefore 9n^2 - \lambda^2 = 0 \quad \dots \quad \dots \quad \dots \quad (8)$$

Hence

$$v^2 = -(n^2 + \lambda^2)x^2 + 2\lambda^2ax. \quad \dots \quad \dots \quad (9)$$

The value of  $x$  for which the velocity  $v$  is a maximum is found by putting  $\ddot{x} = 0$  in (5), or by writing (9) in the form

$$v^2 = -(n^2 + \lambda^2)\left(x - \frac{\lambda^2a}{n^2 + \lambda^2}\right)^2 + \frac{\lambda^4a^2}{n^2 + \lambda^2}.$$

$$\text{We thus obtain } x = \frac{\lambda^2a}{n^2 + \lambda^2}.$$

$$= \frac{9n^2}{10n^2}a, \quad \text{from (8)}$$

$$= \frac{9}{10}a.$$

**94. Motion of a Body when the Mass moving is a Variable Quantity.** Newton's Second Law is that the rate of change of momentum is proportional to the acting force, and up to the present, its mathematical form has been

$$P = mf.$$

This equation however is only true on the assumption that the mass is constant. If the mass is a variable then the law will take the form

$$P = \frac{d}{dt}(mv) = m \frac{dv}{dt} + v \frac{dm}{dt}.$$

**Ex. V.** A heavy uniform chain of length  $a$  and mass  $M$  hangs vertically above a horizontal table, its lower end just touching the table. If it be allowed to fall freely, prove that the thrust on the table at any moment during the fall is three times the weight of the portion on the table.

The thrust on the table consists of

- (1) The weight of the coil on the table,
- (2) The thrust which is due to the destruction of momentum.

Suppose that a length  $x$  is coiled up on the table. Since the chain is falling freely under gravity, the velocity of the chain is then  $\sqrt{2gx}$ . In a small interval of time  $\delta t$ , the length which reaches the table is  $\delta t \sqrt{2gx}$ .



$$\therefore \text{momentum destroyed in time } \delta t = \delta t \cdot \frac{M}{a} \sqrt{2gx} \cdot \sqrt{2gx}.$$

$$\therefore \text{rate of destruction of momentum} = \frac{M}{a} \cdot 2gx.$$

$$\begin{aligned} \therefore \text{thrust on the table} &= \frac{M}{a} \cdot 2gx + \frac{Mg}{a} \cdot x \\ &= \frac{3Mg}{a} x \\ &= \text{three times the weight of the coil} \\ &\quad \text{on the table.} \end{aligned}$$

**Ex. VI.** The total mass of a rocket is 1 lb., including 4 oz. of powder which discharges at the rate of 1 oz. per second at 800 ft./sec. relative to the rocket. Neglecting air resistance, find the greatest height it can attain, starting from rest at ground level.

Let  $v$  ft./sec.  $\equiv$  vel. of rocket after  $t$  sec.

Then the momentum vertically upwards after  $t$  sec. is

$$\left(1 - \frac{t}{16}\right) v \text{ ft. lb. sec. units.}$$

The momentum of this mass after  $(t + \delta t)$  sec. is

$$\left(1 - \frac{t}{16} - \frac{\delta t}{16}\right) (v + \delta v) - \frac{\delta t}{16} (800 - v) \text{ ft. lb. sec. units;}$$

$\therefore$  the change of momentum is

$$\left(1 - \frac{t}{16}\right) \delta v - \frac{v \delta t}{16} - \frac{\delta v \cdot \delta t}{16} - 50 \delta t + \frac{v \delta t}{16} \text{ ft. lb. sec. units.}$$

$\therefore$  the rate of change of momentum is

$$\left(1 - \frac{t}{16}\right) \frac{dv}{dt} - 50 \text{ ft. lb. sec. units.}$$

The equation of motion is

$$\left(1 - \frac{t}{16}\right) \frac{dv}{dt} - 50 = -32 \left(1 - \frac{t}{16}\right),$$

i.e. 
$$\frac{dv}{dt} = \frac{16 \times 50}{16 - t} - 32;$$

$$\therefore v = 16 \times 50 \log \frac{16}{16 - t} - 32t.$$

Let  $x$  ft.  $\equiv$  ht. of rocket after  $t$  sec. ( $t > 4$ ).

Then 
$$\frac{dx}{dt} = 16 \times 50 \log \frac{16}{16 - t} - 32t,$$

$$\therefore x = 16 \left\{ t(50 - t) - 50(16 - t) \log \frac{16}{16 - t} \right\}.$$

When  $t = 4$ ,  $x = 181$  (to 3 sig. figs.) and  $v = 102$  (to 3 sig. figs.).

$$\therefore \text{greatest height} = \left(181 + \frac{102 \times 102}{2 \times 32}\right) \text{ ft.} = 344 \text{ ft.}$$

## Problems XIII.

1. In a particular type of motion the relation

$$\frac{2t}{s} = \frac{1}{v} + \frac{1}{u}$$

exists, where  $s = 0$  and  $v = u$  at  $t = 0$ . Obtain a relation connecting

$s$ ,  $v$  and  $\frac{dv}{ds}$ ; and deduce that

$$\frac{1}{v} = \frac{1}{u} + ks,$$

where  $k$  is a constant. Show also that the retardation is equal to  $kv^3$ .

2. Two equal weights are fastened to the extremities of a string and are then hung over two small smooth pulleys A, B, which are in the same horizontal line. If a third equal weight be fastened at the middle of the horizontal portion AB of the string, show that it will descend a distance equal to two-thirds of AB, and find the velocity and acceleration when it has descended a distance  $x$ .

3. A particle of mass  $m$  is placed initially at a distance  $d$  from a centre of force the attraction to which varies inversely as the square of the distance; prove that the time of arriving at the centre of force is  $\frac{\pi}{2} \sqrt{\frac{dm}{2F}}$  where  $F$  is the force at distance  $d$ .

4. A chain of length  $a$  is coiled up on a ledge at the top of a rough inclined plane, and one end is allowed to slide down. Prove that, if the inclination of the plane be double the angle of friction ( $\lambda$ ),

the chain will be moving freely at the end of time  $\sqrt{\frac{6a}{g}} \cot \lambda$ .

5. AB is smooth straight groove of length  $a$ . A particle moves from rest at A under an attractive force from P which lies on the right bisector of AB at a distance  $d$ . Assuming the attractive force to

be  $\frac{k}{x^2}$ , where  $x$  is the distance of the particle from P, find the velocity of the particle at the middle point of AB. Also find the velocity in the case where the attractive force is  $\mu x$ .

6. A particle moves in a straight line under an attractive force  $\mu^2 x$  to a centre in that straight line; if it be initially at a distance  $c$  from the centre of force and be projected in the straight line with velocity  $V$ , show that it will arrive at the origin position in a time equal to

$$\frac{1}{\mu} \sin^{-1} \frac{\mu c}{\sqrt{V^2 + \mu^2 c^2}}.$$

7. A heavy uniform chain is coiled at the edge of a smooth table and one end is allowed to slip over; prove that it will descend through a space  $\frac{1}{2}gt^2$  in time  $t$  and that its velocity will then be  $\frac{1}{2}gt$ .

8. Prove that, if a body of mass  $m$  be moving under no forces with velocity  $v$ , and the mass of the body be increasing at the rate  $a$ , the equation of motion of the body is  $mv = -va$ .

An open railway truck of weight  $W$  is running on smooth horizontal rails under rain falling vertically which it catches and retains. Show that, if  $V$  is the initial velocity of the truck, the distance travelled in the time  $t$  is

$$(WV/a) \log (1 + at/W),$$

where  $a$  is the weight of rain falling into the truck in the unit of time.

9. A point moves in a straight line under a retardation  $kv^2$ . If its initial velocity be  $u$ , show that the space described in time  $t$  is  $\frac{1}{k} \log_e (1 + kut)$ .

10. A particle is projected in a medium, whose resistance is proportional to the cube of the velocity, and no other forces act on the particle; while the velocity diminishes from  $V_1$  to  $V_2$ , the particle traverses a distance  $d$  in time  $t$ . Show that

$$\frac{d}{t} = \frac{2V_1V_2}{V_1 + V_2}.$$

11. A body is falling from rest in a medium the resistance per unit mass of which is equal to  $k$  times the cube of the velocity. Prove that the time taken for the body to acquire one half its terminal velocity is approximately  $1\frac{3}{8}$  seconds if  $k = g \times 10^{-6}$ .

12. A body falls from rest in a medium the resistance of which is proportional to the velocity. If the terminal velocity is 12 feet per second, find the distance fallen and the velocity acquired after 2 seconds.

13. A body is projected upwards with a velocity of 50 feet per second. If the resistance of the air is assumed to vary as the square of the velocity and the terminal velocity is 150 feet per second, determine the greatest height reached, the time it will take to return to its starting point, and its velocity when it reaches its starting point.

14. A body has a terminal velocity of 16 feet per second in a medium the resistance of which is proportional to the square of the velocity. If it falls from rest, how long will it take to acquire a velocity of 15.8 feet per second?

15. A particle of mass  $M$  is attached to an elastic cord of mass  $m$ , and length  $l$ , and executes small oscillations about its equilibrium position. Prove that its period of oscillation is  $2\pi \sqrt{\frac{M + m/3}{\mu}}$  where  $\mu x$  is the restoring force per unit mass at distance  $x$ .

16. A particle of mass  $m$  hangs vertically at rest at the lower end of a light elastic string, which is fastened at its other end and stretches to twice its natural length. The particle is then acted upon by a vertically downward force, equal to  $\frac{3mg}{2} \sin \mu t$ , whose

period is half that of the free vertical oscillations of the particle. Find when and where the particle first comes to rest again.

17. A particle of mass  $m$  is attached by means of a string of unstretched length  $l$  to a fixed point  $P$  on a horizontal plane, the coefficient of friction of which is  $\mu$ . The particle is projected horizontally from  $P$  with velocity  $\sqrt{(5l\mu g)}$  and the modulus of elasticity of the string is  $\mu mg$ . Prove that the particle will first come to rest when it has reached a distance  $2l$  from  $P$  and find its subsequent motion.

18. Prove that a simple pendulum of length  $l$  can make small oscillations in the same period as a simple pendulum of length  $l'$ , provided the point of suspension be made to execute a horizontal simple harmonic motion of the required period.

19. A particle of mass 8 lbs. is suspended from a light elastic string which stretches one and a half inches under a tension of 4 lbs. weight. When the particle is at rest it is acted on by a vertically downward force of  $0.4 \sin 40t$  pounds weight. Find the displacement of the particle from its initial position after time  $t$  seconds.

20. A chain whose density varies as the distance from an end  $A$  is collected into a heap on a horizontal table; to  $A$  is attached a fine string which, after passing over a small smooth peg vertically above the chain, carries a weight equal to that of a length  $l$  of the chain measured from  $A$ . Prove that a length  $5\frac{1}{2}l$  of chain will be raised from the table before the weight comes to rest.

21. A spherical rain-drop of radius  $a$  cms. falls from rest through a vertical height  $h$ , receiving throughout the motion an accumulation of condensed vapour at the rate of  $k$  gr. per sq. cm. per sec., no vertical force but gravity acting; show that when it reaches the ground its radius will be

$$k\left(\frac{2h}{g}\right)^{\frac{1}{2}}\left\{1 + \left(1 + \frac{ga^3}{2hk^3}\right)^{\frac{1}{2}}\right\}.$$

22. A weight suspended at the end of a light inelastic string 6 feet long is oscillating in a medium the resistance of which per unit mass is  $k$  times the velocity, where  $k$  is a constant. If the amplitude of the vibration is reduced from six inches to four inches in 48 seconds, find the value of  $k$ .

## CHAPTER XIV

### MOTION IN TWO DIMENSIONS : CENTRAL ORBITS

**95. Motion of a Particle in a Medium the Resistance of which is proportional to the Velocity.**

Let  $v$  be the velocity of the particle at any time  $t$ ; then the horizontal and vertical components of the velocity are

$v \frac{dx}{ds}, v \frac{dy}{ds}$  respectively, the  $x$  axis being horizontal and the  $y$  axis vertical.

But 
$$v = \frac{ds}{dt}$$

$\therefore v \frac{dx}{ds} = \frac{ds}{dt} \cdot \frac{dx}{ds} = \frac{dx}{dt}$

and 
$$v \frac{dy}{ds} = \frac{ds}{dt} \cdot \frac{dy}{ds} = \frac{dy}{dt}$$

Let the resistance to the motion of the particle be  $kv$  per unit mass, then the resistances in the horizontal and vertical directions are  $k\dot{x}, k\dot{y}$  respectively, per unit mass.

The equations of motion are therefore

$$\ddot{x} = -k\dot{x} \quad \dots \quad \dots \quad \dots \quad (1)$$

and 
$$\ddot{y} = -g - k\dot{y}. \quad \dots \quad \dots \quad \dots \quad (2)$$

If  $u$  and  $v$  be the initial horizontal and vertical components of the velocity, we have, on integrating (1),

$$\dot{x} + kx = u,$$

i.e., 
$$\frac{dt}{dx} = \frac{1}{u - kx},$$

from which 
$$t = -\frac{1}{k} \log(u - kx) + C.$$

If  $t = 0$  when  $x = 0$ ,  $C = \frac{1}{k} \log u.$

so that 
$$t = \frac{1}{k} \log \frac{u}{u - kx},$$

i.e., 
$$x = \frac{u}{k} \left( 1 - e^{-kt} \right). \quad \dots \quad \dots \quad (3)$$

This equation shows that the horizontal displacement cannot exceed  $\frac{u}{k}.$

Differentiating (3) we have

$$\frac{dx}{dt} = ue^{-kt},$$

which shows that the horizontal velocity rapidly diminishes.

Integrating (2) we obtain

$$\dot{y} + ky = v - gt.$$

Integrating again, after multiplying all through by  $e^{kt}$ , we have

$$\begin{aligned} ye^{kt} &= \int e^{kt} (v - gt) dt \\ &= \frac{v}{k} e^{kt} - \frac{gt}{k} e^{kt} + \frac{g}{k^2} e^{kt} + C. \end{aligned}$$

If we put in the initial conditions this becomes

$$y = \frac{v}{k} - \frac{gt}{k} + \frac{g}{k^2} - \left( \frac{v}{k} + \frac{g}{k^2} \right) e^{-kt}. \quad \dots \quad (4)$$

Differentiating (4) we find the vertical velocity after time  $t$  to be

$$\dot{y} = -\frac{g}{k} + \left( v + \frac{g}{k} \right) e^{-kt} \quad \dots \quad (5)$$

Equation (5) shows that the particle reaches the highest point of its trajectory after a time  $t$ , given by

$$\left( v + \frac{g}{k} \right) = \frac{g}{k} e^{kt},$$

$$\therefore t = \frac{1}{k} \log \frac{kv + g}{g} \quad \dots \quad (6)$$

For greater values of  $t$ , the velocity is downwards; also when the time becomes infinitely great, the velocity will be  $\frac{g}{k}$ .

By substituting the value of  $t$  from (6) in (3), we obtain the horizontal displacement of the particle at the highest point of its path. This is found to be

$$\frac{uv}{kv + g}.$$

### 96. Central Orbits.

A central force is a force acting towards a fixed point and whose intensity is some function of the distance from that point. Given the nature of this force the problem will be to find the path or orbit of a body acted on by the force, and given the orbit, to find the central force.

*To find the differential equation of the path of a body under the action of an attractive force to a fixed centre.*

If a particle is moving in a plane curve, its component accelerations along and perpendicular to the radius vector at any instant are  $\ddot{r} - r\dot{\theta}^2$  and  $2\dot{r}\dot{\theta} + r\ddot{\theta}$  respectively, the values found in Art. 21.

Hence if  $mP$  be the attractive force acting on the body, of mass  $m$ , and if the centre be taken as the origin, the equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = -P \quad \dots \quad (1)$$

and  $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0. \quad \dots \quad (2)$

Now  $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r} \frac{d}{dt} (r^2\dot{\theta})$ , and therefore on integrating

(2) we have

$$r^2\dot{\theta} = h, \quad \dots \quad (3)$$

where  $h$  is a constant.

Also  $\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{h}{r^2} \frac{dr}{d\theta}$ , from (3).  $\dots \quad (4)$

Writing  $u = \frac{1}{r}$ , so that  $\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$ , the last equation becomes

$$\frac{dr}{dt} = -h \cdot \frac{du}{d\theta}.$$

Differentiating,  $\frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt}$   
 $= -h^2 u^2 \frac{d^2u}{d\theta^2}, \quad \text{from (3).}$

Substituting in equation (1),

$$-h^2 u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} \cdot h^2 u^4 = -P$$

or  $\frac{P}{h^2 u^2} = \frac{d^2u}{d\theta^2} + u. \quad \dots \quad (5)$

This is the differential equation of the orbit described. Now  $P$  is, by definition, a central force, and therefore will be given as a function of  $r$ , i.e. of  $u$ . On substituting in (5) the value of  $P$  so given and integrating, the resulting equation will be the polar equation of the orbit. Again, if the equation of the path be given, it will be necessary to find its polar equation referred to the centre as pole, and then by eliminating  $\theta$ , from this equation and equation (5), we shall obtain the force  $P$  in terms of  $u$ , and therefore of  $r$ .

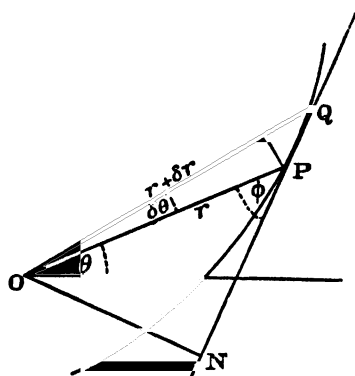


Fig. 147.

Let O be the centre of force, and PQ an element  $\delta s$  of the orbit.

Then  $\delta A = \text{Area OPQ} = \frac{1}{2} r^2 \delta \theta$

$$\therefore \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}.$$

or, from (3)  $= \frac{1}{2} h$ ; (6)

therefore the rate at which the area is being traced out by the radius vector is constant.

Integrating (6) we have

$$A = \frac{1}{2} h t \quad \dots (7)$$

so that the area swept out is proportional to the time.

If  $p$  be the perpendicular from O on the tangent at P, then

$$\delta A = \frac{1}{2} p \delta s$$

$$\therefore \frac{dA}{dt} = \frac{1}{2} p \frac{ds}{dt} \quad \dots \dots \dots (8)$$

But  $\frac{ds}{dt}$  is the velocity  $v$  of the particle in its path so that from (6) and (8)

$$pv = h, \quad \dots \dots \dots (9)$$

which shows that the velocity at any point is inversely proportional to the perpendicular from the centre of force on to the tangent to the path at that point.

Again  $p = r \sin \phi$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi = \frac{1}{r^2} (1 + \cot^2 \phi).$$

But  $\tan \phi = r \frac{d\theta}{dr}$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2} \left( \frac{dr}{d\theta} \right)^2.$$

Putting  $r = \frac{1}{u}$ , so that  $\frac{dr}{d\theta} = -\frac{1}{u^2} \cdot \frac{du}{d\theta}$ , this becomes

$$\frac{1}{p^2} = u^2 + \left( \frac{du}{d\theta} \right)^2. \quad \dots \dots \dots (10)$$

$$\text{Hence, from (9), } v^2 = \frac{h^2}{p^2} = h^2 \left\{ u^2 + \left( \frac{du}{d\theta} \right)^2 \right\} \dots \dots (11)$$

Another expression for the velocity, and from it another differential equation of the path can be found by using the tangential acceleration.



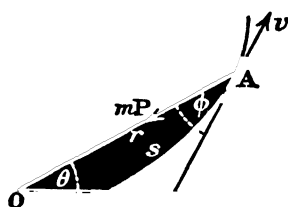


Fig. 148.

Let the velocity of the particle at A be  $v$ . The force  $mP$  acting on the particle is along AO, and is, by hypothesis, a function of  $r$  only. Resolve the force at A along the tangent.

The tangential acceleration is  $\ddot{s}$ , where  $s$  is the distance from some fixed point on the curve,

so that  $\ddot{s} = -P \cos \phi \quad \dots \quad (12)$

But  $\ddot{s} = v \frac{dv}{ds} \quad \text{and} \quad \cos \phi = \frac{dr}{ds}$

$\therefore v \frac{dv}{ds} = -P \frac{dr}{ds} \quad \dots \quad (13)$

Integrating with respect to  $s$ , we have

$$v^2 - v_0^2 = -2 \int_{r_0}^r P dr, \quad \dots \quad (14)$$

where the initial velocity is  $v_0$  and the initial radius is  $r_0$ . Equation (14) shows that the velocity at any point of the orbit is independent of the path described. Moreover, if the velocity at any point is given, this equation enables us to determine the velocity at any other point.

Another differential equation for the orbit can be obtained from (14), for

$$v^2 = \frac{h^2}{p^2}$$

so that  $\frac{h^2}{p^2} = -2 \int P dr + C \quad \dots \quad (15)$

Differentiating with respect to  $r$ ,

$$-\frac{2h^2}{p^3} \cdot \frac{dp}{dr} = -2P$$

or  $P = \frac{h^2}{p^3} \frac{dp}{dr}, \quad \dots \quad (16)$

an expression which is often useful.

The differential equation (16) could also have been obtained from (5), by means of the relation found in (10), viz.,

$$\frac{1}{p^2} = u^2 + \left( \frac{du}{d\theta} \right)^2.$$

Differentiating with respect to  $r$ ,

$$\begin{aligned} -\frac{2}{\dot{p}^3} \frac{d\dot{p}}{dr} &= {}^2u \frac{du}{d\theta} \frac{d\theta}{dr} + {}^2\frac{du}{d\theta} \frac{d\theta}{dr} \frac{d^2u}{d\theta^2} \\ &= {}^2\frac{du}{d\theta} \frac{d\theta}{dr} \left( u + \frac{d^2u}{d\theta^2} \right) \end{aligned}$$

Now  $u = \frac{1}{r} \therefore \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta},$

and  $\therefore \frac{du}{d\theta} \cdot \frac{d\theta}{dr} = -u^2$ .

Hence  $\frac{1}{p^3} \frac{dp}{dr} = u^2 \left( u + \frac{d^2u}{d\theta^2} \right),$

or, using (5),  $\quad = \frac{P}{h^2},$

$$\therefore P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}.$$

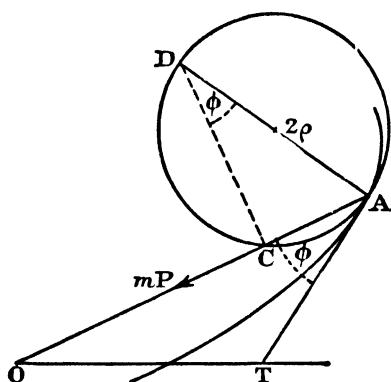


Fig 149.

Let ACD be the circle of curvature at any point A of the curve, and let the radius vector OA cut the circle again at C.

Let  $AD = 2\rho$ ;  $AC = q$ .

If AT is the tangent at A, then  $\widehat{ADC} = \widehat{OAT} = \phi$  and  $q = 2\rho \sin \phi$ . (I7)

Now the normal acceleration of the particle moving with velocity  $v$  at A is  $\frac{v^2}{\rho}$ , and the force acting along the normal is  $mP \sin \phi$ .

$$\therefore \frac{v^2}{\rho} = P \sin \phi$$

$$\therefore v^2 = P_\rho \sin \phi$$

$$= P_{\rho} \cdot \frac{q}{2\rho}, \quad \text{from (17),}$$

$$\therefore \quad v^2 = \frac{1}{2} Pq \quad \dots \quad \dots \quad \dots \quad (18)$$

### 97. Solution of Problems.

1. If the force is given, equation (5) or (16) can be used to obtain the equation of the orbit.

2. If the orbit is given, equation (5) or (16) can be used to obtain the force

3. To find the velocity when the path is not required equation (14) is suitable.

4. To find the velocity when the path is given, equation (18) can be conveniently used.

5. To find the time, the path must be known.

Using equation (3), i.e.,  $r^2 \frac{d\theta}{dt} = h$ ,

we have that 
$$t = \frac{1}{h} \int r^2 d\theta.$$

**98. Motion of a Particle under the Action of a Force to a Fixed Point varying inversely as the Square of the Distance.** This is the law of gravitational attraction.

We have

$$P = \frac{\mu}{r^2} = \mu u^2$$

so that 
$$\frac{d^2 u}{d\theta^2} + u = \frac{P}{h^2 u^2} = \frac{\mu}{h^2} \quad \dots \quad (1)$$

The integral of (1) is (see art. 93 I.)

$$u = \frac{\mu}{h^2} + A \cos(\theta + \alpha) \quad \dots \quad (2)$$

$$= \frac{\mu}{h^2} \{1 + e \cos(\theta + \alpha)\} \quad \dots \quad (3)$$

where 
$$e = \frac{h^2 A}{\mu}.$$

The polar equation of a conic section referred to the focus as origin is

$$lu = 1 + e \cos(\theta + \alpha), \quad \dots \quad (4)$$

where  $e$  is the eccentricity,  $\alpha$  is the angle which the initial line makes with the major axis, and  $2l$  is the latus rectum.

Comparing (3) and (4) we see that the orbit is a conic section, the focus being at the centre of force, and that

$$l = \frac{h^2}{\mu}.$$

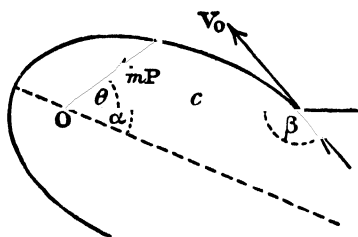


Fig. 150

The nature of the conic section will depend on the initial conditions.

Suppose initially that  $\theta = 0$ ,  $r = c$ ,  $v = v_0$  and that the particle is projected at an angle  $\beta$  to the initial radius vector  $c$ .

Now  $\tan \phi = r \frac{d\theta}{dr} = -u \frac{d\theta}{du} \dots \dots \dots (5)$

Differentiating (2)

$$\frac{du}{d\theta} = -A \sin(\theta + \alpha).$$

But when  $\phi = \beta, u = \frac{1}{c}$  and  $\theta = 0$

$$\therefore \tan \beta = + \frac{1}{cA \sin \alpha}.$$

To simplify our equations, suppose also that  $\beta = 90^\circ$   
then  $\cot \beta = + cA \sin \alpha = 0$

$$\therefore \alpha = 0.$$

Again, from (2)

$$\begin{aligned} \frac{1}{c} &= A \cos \alpha + \frac{\mu}{h^2} \\ &= A + \frac{\mu}{h^2}, \end{aligned}$$

$$\therefore A = \frac{1}{c} - \frac{\mu}{h^2} \dots \dots \dots (6)$$

But if the direction of motion is initially at right angles to the radius vector

$$p = c \text{ and as } v = \frac{h}{p},$$

$$\therefore h = v_0 c.$$

Substituting this value of  $h$  in (6), we have

$$A = \frac{1}{c} - \frac{\mu}{v_0^2 c^2}.$$

Then

$$\begin{aligned} e &= \frac{h^2}{\mu} A = \frac{v_0^2 c^2}{\mu} A \\ &= \frac{v_0^2 c}{\mu} - 1. \dots \dots (7) \end{aligned}$$

Hence if  $\frac{v_0^2 c}{\mu} > 2$ , i.e., if  $v_0^2 > \frac{2\mu}{c}$ , the curve will be a *hyperbola*, since  $e > 1$ .

Similarly, if  $v_0^2 = \frac{2\mu}{c}$  the curve will be a *parabola*, since  $e = 1$ ,

and if  $v_0^2 < \frac{2\mu}{c}$  the curve will be an *ellipse*, since  $e < 1$



Hence, if

$v_0^2 > \frac{2\mu}{c}$ , then an infinite distance may be reached,

i.e., the orbit will be a hyperbola.

Similarly, if  $v_0^2 = \frac{2\mu}{c}$ , then an infinite distance will just be reached, so that the orbit will be a parabola, and if  $v_0^2 < \frac{2\mu}{c}$ , then an infinite distance cannot be reached, so that the orbit, being closed, must be an ellipse.

In the case of the circle  $v_0^2 = \frac{\mu}{c} =$  one-half the square of the velocity due to a fall from infinity.

### 100. The Law of the Inverse Square using the Expression

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}.$$

Now

$$P = \frac{\mu}{r^2}$$

$$\therefore \frac{\mu}{r^2} = \frac{h^2}{p^3} \cdot \frac{dp}{dr}.$$

$$\text{Integrating, } v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad \dots \quad (1)$$

which is the same expression as (2) of the previous article.

Now the equation of a conic referred to the focus as pole is

$$\frac{b^2}{p^2} = \frac{2a}{r} \begin{cases} + 1, \text{ for the hyperbola} \\ - 1, \text{ for the ellipse} \\ + 0, \text{ for the parabola.} \end{cases} \quad \dots \quad (2)$$

Hence (1) represents a hyperbola, ellipse or parabola according as C is positive, negative or zero.

If  $v_0$  is the velocity of projection when  $r = c$

$$v_0^2 = \frac{2\mu}{c} + C \quad \dots \quad (3)$$

$$\therefore C = v_0^2 - \frac{2\mu}{c},$$

an expression which determines whether C is positive, negative or zero.

$$\text{If } C = 0, \text{ then } v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r}$$

$$\therefore v = \sqrt{\frac{2\mu}{r}} \quad \dots \quad (4)$$

This expression shows that the kinetic energy of a particle describing a parabolic path is inversely proportional to the distance of the particle from the focus.

If  $C$  is not equal to 0, then, equating co-efficients in (1) and (2),

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \pm C,$$

$$\therefore h^2 = \frac{b^2 \mu}{a}.$$

$$\begin{aligned} \text{Also } v^2 &= \frac{h^2}{p^2} = \frac{b^2 \mu}{p^2 a} = \frac{\mu}{a} \left( \frac{2a}{r} \pm 1 \right), \quad \text{from (2),} \\ &= \mu \left( \frac{2}{r} \pm \frac{1}{a} \right). \quad \dots \dots (5) \end{aligned}$$

If  $2l$  is the latus rectum,

$$\text{then } 2l = \frac{2b^2}{a} = \frac{2h^2}{\mu}.$$

It will be seen from (4) and (5) that in each case the velocity in the orbit is independent of the direction of the velocity.

**Ex. I.** A particle of mass  $m$  is projected under the central force  $m\mu/r^2$ . If it be projected from the point  $r = c$ ,  $\theta = 0$  at right angles to the radius, with a velocity  $(n\mu/c)^{\frac{1}{2}}$ , prove that

$$\left( r \frac{dr}{dt} \right)^2 = \frac{\mu}{c} (r - c) \{ nc - r(2 - n) \}.$$

The equations of motion are,

$$r^2 \ddot{\theta} = h, \quad \dots \dots \dots (1)$$

$$\text{and } \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}. \quad \dots \dots (2)$$

When  $r = c$ ,  $\theta = 0$  and if  $\dot{\theta}_0$  be the initial angular velocity

$$\text{then } c^2 \dot{\theta}_0 = h. \quad \dots \dots \dots (3)$$

But the particle is projected at right angles to the initial line with velocity  $\sqrt{\frac{n\mu}{c}}$ ,

$$\therefore \sqrt{\frac{n\mu}{c}} = c\dot{\theta}_0,$$

$$\therefore \dot{\theta}_0 = \frac{1}{c} \sqrt{\frac{n\mu}{c}},$$

$$\therefore \text{from (3)} \quad \frac{h}{c^2} = \frac{1}{c} \sqrt{\frac{n\mu}{c}};$$

$$\text{and } \therefore h = \sqrt{n\mu c}. \quad \dots \dots \dots (4)$$

Substituting for  $\dot{\theta}$  from (1) in (2) we have

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^2}. \quad \dots \dots \dots (5)$$

Multiplying all through by  $2\dot{r}$  and integrating, we obtain,

$$\dot{r}^2 + \frac{h^2}{r^2} = \frac{2\mu}{r} + A. \quad \dots \dots \dots (6)$$

Now initially the particle is moving at right angles to the radius vector

$$\therefore \dot{r} = 0 \quad \text{when } r = c.$$

so that

$$\begin{aligned} A &= \frac{h^2}{c^2} - \frac{2\mu}{c}; \text{ or from (4),} \\ &= \frac{n\mu}{c} - \frac{2\mu}{c} = \frac{(n-2)\mu}{c}; \quad \dots \dots (7) \end{aligned}$$

hence (6) becomes

$$\begin{aligned} \left(r \frac{dr}{dt}\right)^2 &= 2\mu r + \frac{(n-2)\mu r^2}{c} - n\mu c \\ &= \frac{\mu}{c} \left[ r^2(n-2) + 2cr - nc^2 \right] \\ &= \frac{\mu}{c} \left\{ nc - r(2-n) \right\} (r-c). \end{aligned}$$

Using the conditions of art. 98 or art. 99, we see that the path is a hyperbola, parabola or ellipse according as  $n$  is greater than, equal to, or less than 2. In  $n = 1$ , the path is circular.

**Ex. II.** A particle is projected with velocity  $\sqrt{6\mu/5c}$  from a point at distance  $c$  from a centre of force  $\mu/r^2$  in a direction making an angle  $\sin^{-1}\sqrt{\frac{2}{3}}$  with the radius. Show that its greatest and least distances from the centre of force are  $2c$  and  $\frac{1}{2}c$ .

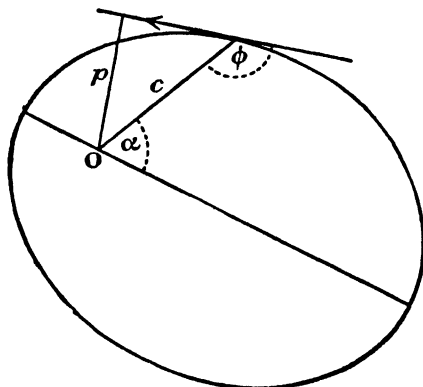


Fig. 152.

Again  $h = v\rho = v_0\rho_0$ , where  $v_0$  and  $\rho_0$  are the initial values of  $v$  and  $\rho$ .

Now  $v_0 = \sqrt{\frac{6\mu}{5c}}$  and  $\rho_0 = c \sin \phi_0$ , where  $\phi_0$  is the initial angle

Since

$$u + \frac{d^2u}{d\theta^2} = \frac{\mu}{h^2}$$

the equation of the curve is

$$u = \frac{\mu}{h^2} \left\{ 1 + e \cos(\theta + \alpha) \right\} \quad (1)$$

The initial conditions will enable us to determine the unknowns  $h$ ,  $e$  and  $\alpha$ .

Initially  $u = \frac{1}{c}$  and  $\theta = 0$ .

Substituting these values in (1)

$$\frac{1}{c} = \frac{\mu}{h^2} \left\{ 1 + e \cos \alpha \right\} \quad \dots (2)$$



between the tangent to the curve and the radius vector.

$$\therefore \quad p_0 = c\sqrt{\frac{2}{3}}$$

$$\text{and} \quad h = \sqrt{\frac{6\mu}{5c}} \cdot c \sqrt{\frac{2}{3}} = 2\sqrt{\frac{\mu c}{5}}$$

$$\therefore \quad \frac{\mu}{h^2} = \frac{5}{4c} \dots \dots \dots (4)$$

Substituting in (2)

$$\begin{aligned} \frac{1}{r} &= 1 + e \cos \alpha \\ \text{i.e.,} \quad e \cos \alpha &= -\frac{1}{2}, \quad \dots \dots \dots (5) \end{aligned}$$

and hence  $\alpha$  lies between  $\frac{\pi}{2}$  and  $\pi$ .

Again

$$\tan \phi = -u \frac{d\theta}{du} \dots \dots \dots (6)$$

$$\text{Initially} \quad \tan \phi = -\sqrt{2}, \quad u = \frac{1}{c}.$$

$$\text{Differentiating (1)} \quad \frac{du}{d\theta} = -\frac{\mu e}{h^2} \sin(\theta + \alpha).$$

$$\text{Initially therefore} \quad \frac{d\theta}{du} = -\frac{4c}{5e \sin \alpha} \quad \text{since } \theta = 0$$

$$\text{and equation (6) becomes} \quad \sqrt{2} = \frac{1}{c} \frac{4c}{5e \sin \alpha},$$

$$\text{from which} \quad e^2 \sin^2 \alpha = \frac{8}{25} \dots \dots \dots (7)$$

$$\begin{aligned} \text{From (5) and (7)} \quad e &= \frac{2}{5} \dots \dots \dots (8) \\ \text{so that the curve (1) is an ellipse.} \end{aligned}$$

Now  $v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right)$ , where  $a$  is the semi-major axis of the ellipse,

$$\text{so that} \quad v_0^2 = \frac{6\mu}{5c} = \mu \left( \frac{2}{c} - \frac{1}{a} \right)$$

$$\text{from which} \quad a = \frac{5c}{4} \dots \dots \dots (9)$$

The pole is one of the foci of the ellipse, and the maximum distances of the particle from the centre of force are therefore those at which the particle is at the ends of the major axis.

$$\text{These distances are} \quad a(1 + e) = a \left( 1 + \frac{2}{5} \right) = \frac{8a}{5}$$

$$\text{and} \quad a(1 - e) = a \left( 1 - \frac{2}{5} \right) = \frac{2a}{5}.$$

But  $a = \frac{5c}{4}$ ;

the required distances are therefore  $2c$  and  $\frac{1}{2}c$ .

**101 (a). To determine the Law of Force to the Centre under which a Particle describes an Ellipse.**

The tangential-polar equation of the ellipse, referred to its centre as pole, is

$$\frac{a^2 b^2}{p^3} = a^2 + b^2 - r^2. \quad \dots \quad (1)$$

Differentiating with respect to  $r$ , we have

$$\frac{a^2 b^2}{p^3} \cdot \frac{dp}{dr} = r.$$

Now

$$P = \frac{h^2}{p^3} \frac{dp}{dr},$$

$$\therefore P = \frac{h^2}{p^3} \cdot \frac{r p^3}{a^2 b^2} = \frac{h^2}{a^2 b^2} \cdot r,$$

and therefore the force varies as the distance.

**(b). To determine the Law of Force to the Focus under which a Particle describes an Ellipse.**

The equation of the ellipse is

$$lu = 1 + e \cos \theta.$$

$$\therefore l \frac{du}{d\theta} = -e \sin \theta,$$

$$\therefore \frac{d^2 u}{d\theta^2} = -\frac{e}{l} \cos \theta,$$

$$\begin{aligned} \therefore u + \frac{d^2 u}{d\theta^2} &= \frac{1}{l} + \frac{e}{l} \cos \theta - \frac{e}{l} \cos \theta \\ &= \frac{1}{l}. \end{aligned}$$

$$\begin{aligned} \text{Hence } P &= h^2 u^3 \left( u + \frac{d^2 u}{d\theta^2} \right) \\ &= \frac{h^2}{l} \cdot u^3, \end{aligned}$$

so that the force varies inversely as the square of the distance.

$$\text{If } P = \frac{\mu}{r^2}, \text{ then } h = \sqrt{\mu l}.$$

$$\text{Again, } v^2 = \frac{h^2}{p^2} = h^2 \left\{ u^2 + \left( \frac{du}{d\theta} \right)^2 \right\}$$

$$\begin{aligned}
&= \mu l \left\{ \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta}{l^2} + \frac{e^2 \sin^2 \theta}{l^2} \right\} \\
&= \mu \left( 2 \cdot \frac{1 + e \cos \theta}{l} - \frac{1 - e^2}{l} \right) \\
&= \mu \left( \frac{2}{r} - \frac{1}{a} \right),
\end{aligned}$$

for  $l = \frac{b^2}{a} = a(1 - e^2).$

Hence the velocity depends on the distance and not on the direction.

If the particle is projected with velocity  $v_0$  at distance  $c$ , in any direction, the semi-major axis is given by

$$v_0^2 = \mu \left( \frac{2}{c} - \frac{1}{a} \right).$$

### 102. Velocity in an Ellipse when Attraction is towards Centre.

From § 101 (a),  $P = \frac{h^2}{a^2 b^2} r = \mu r \therefore \mu = \frac{h^2}{a^2 b^2} \dots (1)$

But  $v^2 = \frac{h^2}{p^2} = \frac{h^2}{a^2 b^2} (a^2 + b^2 - r^2)$  from § 101, eqn. (1)  
 $= \mu(a^2 + b^2 - r^2) \dots \dots \dots (2)$

Now if CP and CD are conjugate diameters and CP =  $r$ , then

$$CD^2 = a^2 + b^2 - r^2.$$

Hence, from (2),  $v = CD \sqrt{\mu} \dots (3)$

Moreover, since the tangent at P is parallel to CD, the velocity at P is parallel to CD. If the time for a part of the orbit is required, we may make use of the fact that the ellipse can be considered as the projection of a circle.

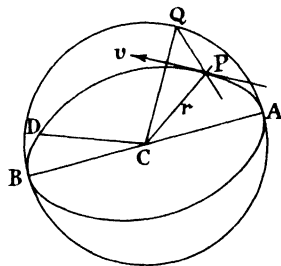


Fig. 153.

The sectoral area ACQ projects into the area ACP, and therefore

$$\frac{\text{Area ACP}}{\text{Area ACQ}} = \frac{b}{a} \dots \dots (4)$$

Since  $\frac{dA}{dt} = \frac{1}{2}h$ ,  $\therefore A = \frac{1}{2}ht$ ,  $\dots \dots (5)$

so that, using (4) and (5)

$$t = \frac{2A}{h} = \frac{2b}{ah} \cdot \text{area ACQ} = \frac{2b}{ah} \frac{1}{2} a^2 \widehat{ACQ}$$

$$= \frac{ab}{h} \widehat{ACQ} = \frac{\widehat{ACQ}}{\sqrt{\mu}} \quad \dots \quad \text{from (1)}$$

Note that  $\widehat{ACQ} = t\sqrt{\mu}$ , and therefore CQ moves with constant angular velocity  $\sqrt{\mu}$ , as P moves in the ellipse.

### 103. Time in an Orbit.

The time is to be found from the equations

$$r^2 \dot{\theta} = h \quad \dots \quad \dots \quad (1)$$

$$\text{or} \quad \frac{dA}{dt} = \frac{1}{2}h. \quad \dots \quad \dots \quad (2)$$

For the periodic time T, the integral of (2) gives

$$A = \frac{1}{2}hT.$$

But the area of an ellipse is  $\pi ab$ .

$$\therefore \quad T = \frac{2A}{h} = \frac{2\pi ab}{h}. \quad \dots \quad \dots \quad (3)$$

If the force is to the **focus** of the ellipse

$$h = \sqrt{\mu l} = \sqrt{\mu \frac{b^2}{a}},$$

$$\therefore \quad T = \frac{2\pi ab}{\sqrt{\mu}} \frac{\sqrt{a}}{b} = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}, \quad \dots \quad \dots \quad (4)$$

where  $a$  is the semi-major axis.

If the force is to the **centre** of the ellipse

$$P = \frac{h^2}{a^2 b^2} r. \quad (\text{Art. 101.})$$

Now  $\mu$  is the force at unit distance

$$\therefore \quad \mu = \frac{h^2}{a^2 b^2}, \text{ i.e., } h = ab\sqrt{\mu}.$$

$$\text{Hence} \quad T = \frac{2A}{h} = \frac{2\pi ab}{ab\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}. \quad \dots \quad \dots \quad (5)$$

The periodic time in this case is therefore independent of the dimensions of the orbit.

### 104. To determine the Time in an Ellipse when the Force is to the Focus.

The equation of the ellipse is

$$\frac{l}{r} = 1 + e \cos \theta \quad \dots \quad \dots \quad (1)$$

where the major axis of the ellipse is the initial line and the pole is at the focus.

Now

$$r^2 \dot{\theta} = h$$

 $\therefore$ 

$$\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{r^2}{\sqrt{\mu l}}$$

$$= \frac{l^{\frac{3}{2}}}{\sqrt{\mu}} \frac{1}{(1 + e \cos \theta)^2} \quad \text{from (1),}$$

 $\therefore$ 

$$t = \frac{l^{\frac{3}{2}}}{\sqrt{\mu}} \int_{\theta_1}^{\theta_2} \frac{d\theta}{(1 + e \cos \theta)^2} \quad \dots \quad (2)$$

$$\begin{aligned} \text{Now } 1 + e \cos \theta &= \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} + e \left( \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) \\ &= \cos^2 \frac{\theta}{2} (1 + e) + \sin^2 \frac{\theta}{2} (1 - e) \end{aligned}$$

$$\therefore \frac{1}{1 + e \cos \theta} = \frac{\sec^2 \frac{\theta}{2}}{(1 + e) + (1 - e) \tan^2 \frac{\theta}{2}} \quad \dots \quad (3)$$

$$\begin{aligned} \therefore \int \frac{d\theta}{1 + e \cos \theta} &= \frac{2}{1 - e} \int \frac{d \left( \tan \frac{\theta}{2} \right)}{\frac{1 + e}{1 - e} + \tan^2 \frac{\theta}{2}} \\ &= \frac{2}{\sqrt{1 - e^2}} \tan^{-1} \left\{ \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\theta}{2} \right\}. \end{aligned}$$

Now let

$$\phi = \tan^{-1} \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\theta}{2}$$

so that

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \phi. \quad \dots \quad (4)$$

Then

$$(1 - e) \tan^2 \frac{\theta}{2} = (1 + e) \tan^2 \phi.$$

$$\begin{aligned} \therefore (1 + e) + (1 - e) \tan^2 \frac{\theta}{2} &= (1 + e)(1 + \tan^2 \phi) \\ &= (1 + e) \sec^2 \phi. \end{aligned}$$

Also differentiating (4),

$$\sec^2 \frac{\theta}{2} \cdot d\theta = 2 \sqrt{\frac{1 + e}{1 - e}} \sec^2 \phi d\phi. \quad \dots \quad (5)$$

From (3), we have

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{(1 + e \cos \theta)^2} = \int_{\theta_1}^{\theta_2} \frac{\sec^2 \frac{\theta}{2} d\theta}{\left\{ (1 + e) + (1 - e) \tan^2 \frac{\theta}{2} \right\}^2}$$

$$\begin{aligned}
&= \frac{2}{(1-e^2)^{\frac{3}{2}}} \int_{\phi_1}^{\phi_2} \frac{(1-e) + (1+e) \tan^2 \phi}{\sec^2 \phi} d\phi \\
&= \frac{2}{(1-e^2)^{\frac{3}{2}}} \int_{\phi_1}^{\phi_2} \{ (1-e) \cos^2 \phi + (1+e) \sin^2 \phi \} d\phi \\
&= \frac{2}{(1-e^2)^{\frac{3}{2}}} \int_{\phi_1}^{\phi_2} (1 - e \cos 2\phi) d\phi. \\
&= \frac{2}{(1-e^2)^{\frac{3}{2}}} \left[ \phi - \frac{e}{2} \sin 2\phi \right]_{\phi_1}^{\phi_2} \\
&= \frac{1}{(1-e^2)^{\frac{3}{2}}} \left[ 2 \tan^{-1} \left\{ \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right\} - \frac{e \sqrt{1-e^2} \sin \theta}{1+e \cos \theta} \right]_{\theta_1}^{\theta_2},
\end{aligned}$$

since

$$\sin 2\phi = \frac{2 \tan \phi}{1 + \tan^2 \phi} = \frac{2 \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2}}{1 + \frac{1-e}{1+e} \tan^2 \frac{\theta}{2}} = \frac{\sqrt{1-e^2} \sin \theta}{1 - e \cos \theta}$$

Also

$$l = a(1-e^2)$$

so that the time from the vertex, where  $\theta_1 = 0$  is given by

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left[ 2 \tan^{-1} \left\{ \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right\} - \frac{e \sqrt{1-e^2} \sin \theta}{1+e \cos \theta} \right].$$

**Ex. III.** A particle describes an elliptic orbit under a central force to one of the foci. The greatest and least velocities in the orbit being 110 ft. per sec. and 90 ft. per sec. respectively, and the periodic time being 20 mins., calculate the eccentricity and (approximately) the length of the major axis.

Now the velocity in the orbit is given by

$$v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right) \quad \dots \quad \dots \quad \dots \quad (1)$$

so that  $v$  is a maximum when  $r$  is a minimum, and  $v$  is a minimum when  $r$  is a maximum.

Again  $v = \frac{h}{p}$ , so that  $v$  varies inversely as  $p$ , and its direction is perpendicular to the radius vector at the extremities of the major axis.

If then  $2a$  is the length of the major axis, and  $e$  the eccentricity,

$$110 = \frac{h}{a(1-e)} \quad \dots \quad \dots \quad \dots \quad (2)$$

and

$$90 = \frac{h}{a(1+e)} \quad \dots \quad \dots \quad \dots \quad (3)$$

Dividing, we obtain

$$\frac{11}{9} = \frac{1+e}{1-e}$$

so that

$$e = \frac{1}{10} \quad \dots \quad \dots \quad \dots \quad (4)$$

If  $T$  is the periodic time

$$T = \frac{2\pi ab}{h}$$

$$\therefore 20 \times 60 = \frac{2 \cdot \pi \cdot a \cdot a(1-e^2)^{\frac{1}{2}}}{110a(1-e)} \quad \dots \quad \dots \quad \dots \quad (5)$$

$$\begin{aligned} \therefore a &= \frac{20 \times 60 \times 110}{2\pi} \left( \frac{1-e}{1+e} \right)^{\frac{1}{2}} \text{ ft.} \\ &= \frac{20 \times 60 \times 110}{2\pi} \cdot \frac{3}{\sqrt{11}} \text{ ft.} \\ &= 3.6 \text{ miles approximately.} \end{aligned}$$

$\therefore$  the major axis has a length of 7.2 miles (approximately).

**Ex. IV.** A particle of mass  $m$  moves under a central force  $m\mu/(\text{distance})^3$ , and is projected at a distance  $c$  from the centre of force with the velocity which would be acquired by a fall from rest at infinity to the point of projection. Prove that the orbit is an equiangular spiral, and that the time elapsed till the particle reaches a distance  $r$  from the centre is

$$\frac{1}{2}(r^2 - c^2) \sec \alpha / \sqrt{\mu};$$

where  $\alpha$  is the angle in the spiral between radius vector and tangent.

Using the expression

$$P = \frac{h^2}{p^3} \frac{dp}{dr},$$

we have

$$\frac{\mu}{r^3} = \frac{h^3}{p^3} \frac{dp}{dr} \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\text{Integrating (1),} \quad \frac{\mu}{r^2} = \frac{h^2}{p^2} + C \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\text{But } v = \frac{h}{p} \therefore \frac{\mu}{r^2} = v^2 + C \quad \dots \quad \dots \quad \dots \quad (3)$$

The velocity from rest at infinity is given by the energy equation

$$\frac{1}{2}mv^2 = m \int_c^\infty \frac{\mu}{r^3} dr$$

$$= \frac{1}{2} m \left[ -\frac{\mu}{r^2} \right]_c^\infty$$

$$= \frac{1}{2} m \frac{\mu}{c^2}$$

so that  $v^2 = \frac{\mu}{c^2} \dots \dots \dots (4)$

Substituting in (3)  $\frac{\mu}{c^2} = \frac{\mu}{c^2} + C$

so that  $C = 0$

and  $\frac{\mu}{r^2} = v^2 = \frac{h^2}{p^2}$

$\therefore p = \frac{h}{\sqrt{\mu}} r \dots \dots \dots (5)$

But  $h$  and  $\mu$  are constants.  $\therefore p$ , varies as  $r$ , which is the property of the equiangular spiral.

Writing  $h = \sqrt{\mu} \sin \alpha$ , in equation (5) we have

$$p = r \sin \alpha, \dots \dots \dots (6)$$

where  $\alpha$  is the angle between the radius vector and the tangent.

Now from the expression

$$\tan \phi = r \frac{d\theta}{dr},$$

where  $\phi$  is the angle between the radius vector and the tangent, we have

$$\tan \alpha = r \frac{d\theta}{dr}.$$

Again  $\frac{dt}{dr} = \frac{dt}{d\theta} \cdot \frac{d\theta}{dr} = \frac{\tan \alpha}{r} \cdot \frac{dt}{d\theta} \dots \dots \dots (7)$

The time in the orbit is found from

$$r^2 \frac{d\theta}{dt} = h$$

or  $\frac{dt}{d\theta} = \frac{r^2}{h} \dots \dots \dots (8)$

Hence using (6), (7) and (8)

$$\frac{dt}{dr} = \frac{\tan \alpha}{r} \cdot \frac{r^2}{\sqrt{\mu} \sin \alpha} = \frac{r}{\sqrt{\mu}} \sec \alpha.$$

$\therefore t = \frac{\sec \alpha}{\sqrt{\mu}} \int_r^r r dr$

$$= \frac{\sec \alpha}{2\sqrt{\mu}} (r^2 - c^2).$$



**105. Apses.**—An **apse** is a point in a central orbit at which the radius vector, for which the centre of force is the pole, is a maximum or a minimum, the corresponding values of the radius vector being called the **apsidal distances**.

Thus the planets, as we know, move round the sun as the centre of force in elliptical orbits. The point in the orbit most distant is called the **aphelion**, and the least distant the **perihelion**, points which when used of the moon's orbit are termed **apogee** and **perigee** respectively.

The analytical condition for a maximum or minimum value of the radius vector is that  $\frac{du}{d\theta}$  shall vanish, and that the first succeeding differential coefficient which does not vanish shall be of an even order.

Now, if  $\phi$  is the angle between the radius and the tangent to the curve

$$\tan \phi = -u \frac{d\theta}{du},$$

so that when  $\frac{du}{d\theta} = 0$ ,  $\phi = \frac{\pi}{2}$ .

Hence, the first condition for an apse ensures that the tangent at an apse is perpendicular to the radius vector.

When the central acceleration is  $\mu u^2$ , the equation of motion is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{\mu}{h^2}.$$

Multiplying by  $h^2 \frac{du}{d\theta}$  and integrating, we have

$$\frac{h^2}{2} \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right] = \mu u + C. \quad \dots (1)$$

The particle is at an apse when  $\frac{du}{d\theta} = 0$ .

Equation (1) then becomes

$$\frac{h^2}{2} u^2 - \mu u - C = 0,$$

which shows that there are only two apsidal distances. These distances are evidently the segments of the major axis of the conic.

It is, however, easy to show that for any central orbit, there are only two different apsidal distances. By hypothesis, the acceleration of the particle in the orbit is a function of the

distance from the centre of attraction. If then, the direction of motion at an apse were reversed, the speed being unaltered, the particle would necessarily describe a path similar to and symmetrical with the path already described, so that the apsidal line divides the orbit into two equal and similar parts. It immediately follows that there can only be two apsidal distances. The angle between two consecutive apsidal distances is called the **apsidal angle**. When the orbit is a conic about a focus the apsidal angle is  $180^\circ$ .

**Ex. V.** Find the law of force to the pole when the path is the cardioid,  $r = a(1 - \cos \theta)$ ; and prove that, if  $Q$  be the force at the apse, and  $V$  the velocity

$$3V^2 = 4aQ.$$

Since  $r = a(1 - \cos \theta) \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$

$\therefore \quad \frac{dr}{d\theta} = a \sin \theta. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$

Now 
$$\begin{aligned} \frac{1}{p^2} &= \frac{1}{r^2} + \frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 \\ &= \frac{1}{r^2} + \frac{1}{r^4} a^2 \left\{ 1 - \left( \frac{a-r}{a} \right)^2 \right\} \\ &= \frac{2a}{r^3}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (3) \end{aligned}$$

Then, since  $P = \frac{h^2}{p^3} \frac{dp}{dr}$ , where  $P$  is the force per unit mass,

and 
$$-\frac{2}{p^3} \cdot \frac{dp}{dr} = -\frac{6a}{r^4}.$$

$\therefore \quad P = \frac{h^2}{p^3} \cdot \frac{3ap^3}{r^4} = \frac{3ah^2}{r^4},$

so that the force is proportional to the inverse fourth power of the distance.

Now at an apse 
$$\frac{du}{d\theta} = 0$$

or 
$$-\frac{1}{r^2} \frac{dr}{d\theta} = 0$$

$\therefore \quad -\frac{1}{r^2} a \sin \theta = 0$

from which either  $\sin \theta = 0$  or  $r$  is infinitely great, the latter case being inadmissible so long as the particle is moving along the cardioid.

Thus  $\theta = \pi$  or  $0$ .

When  $\theta = \pi$ ,  $r = a(1 + 1) = 2a$

and

$$Q = \frac{3ah^2}{r^4} = \frac{3ah^2}{16a^4} = \frac{3h^2}{16a^3}.$$

Also

$$p^2 = \frac{r^3}{2a} = \frac{8a^3}{2a} = 4a^2$$

and

$$V^2 = \frac{h^2}{p^2} = \frac{h^2}{4a^2},$$

so that

$$4aQ = \frac{3h^2}{4a^2} = 3V^2.$$

When  $\theta = 0$ ,  $r = 0$  and  $p = 0$ , and the particle is moving with infinite velocity along the axis of the cardioide, and thus continues to move in a straight line.

**Ex. VI.** If, when a particle is at an end of the minor axis of an ellipse, the intensity of attraction is increased in the ratio 3 : 2, prove that the axes of the new orbit are  $\frac{3a}{2}$ ,  $\sqrt{2}.b$ , where  $2a$ ,  $2b$  are the old axes.

The velocity in the orbit is given by

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

Hence, when the particle is at one extremity of the minor axis

$$v^2 = \frac{2\mu}{a} - \frac{\mu}{a} = \frac{\mu}{a}. \quad \dots \quad \dots \quad \dots \quad (1)$$

Let  $\mu_1$  be the new intensity of attraction,  $2a_1$  and  $2b_1$  the new axes.

Now the velocity when  $r = a$  is unaltered.

$$\therefore v^2 = \frac{2\mu_1}{a} - \frac{\mu_1}{a_1} = \frac{\mu}{a} \quad \text{from (1).}$$

But  $\mu_1 = \frac{3}{2}\mu$

$$\therefore \frac{3}{a} - \frac{3}{2a_1} = \frac{1}{a}$$

$$\therefore 2a_1 = \frac{3a}{2}. \quad \dots \quad \dots \quad \dots \quad (2)$$

Now

$$h = vp.$$

But  $v$  is unaltered in magnitude and direction, so that the constant  $h$  is the same for both orbits.

But

$$h = \sqrt{\mu \frac{b^2}{a}}.$$

$$\therefore \sqrt{\mu \frac{b^2}{a}} = \sqrt{\frac{3}{2}\mu \cdot \frac{b_1^2}{a_1}}$$

$$\therefore \frac{b^2}{a} = \frac{3}{2} \frac{b_1^2}{a_1}$$

$$\therefore \frac{b_1^2}{b^2} = \frac{2}{3} \frac{a_1}{a} = \frac{1}{2}, \text{ from (2)}$$

$$\therefore 2b_1 = \sqrt{2} \cdot b.$$

**Ex. VII.** A particle describes an ellipse under a force to the focus. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without change in the direction of motion. Prove that the particle proceeds to describe a parabola whose latus rectum is double that of the ellipse.

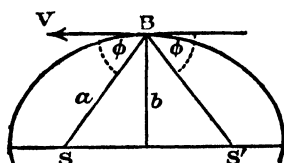


Fig. 154.

Let S be the centre of force, one focus of the ellipse, and let S' be the other focus.

Let B be the extremity of the minor axis so that  $SB = a$ , where  $2a$  is the length of the major axis.

The velocity at B is given by

$$v^2 = \frac{2\mu}{a} - \frac{\mu}{a}$$

$$= \frac{\mu}{a}.$$

The kinetic energy is doubled, but the attractive force is unaltered, so that the velocity is then given by

$$v^2 = \frac{2\mu}{a}.$$

But this is the velocity of a particle describing a parabolic path at distance  $a$  from the centre of force. The particle therefore proceeds to describe a parabola about S as the focus.

Since the direction of motion is un-

altered, BS' is parallel to the axis of the parabola.

Let T be a point on the tangent at B such that  $SB = ST$ .

Then ST, which is parallel to BS', is the axis of the parabola.

From S draw SY perpendicular to BT and from Y draw YA perpendicular to ST, meeting ST at A.

Then YA is the tangent at the vertex A of the parabola.

The polar equation of the parabola is

$$\frac{l}{r} = 1 + \cos \theta,$$

where  $2l$  is the latus rectum and

where

$$\theta = \angle TSB$$

$\therefore$

$$\theta = 180 - 2\phi,$$

and

$$l = a(1 + \cos 180 - 2\phi)$$

$$= a(1 - \cos 2\phi)$$

$$= 2a \sin^2 \phi.$$

But, referring to figure 154,

$$\sin \phi = \frac{b}{a}.$$

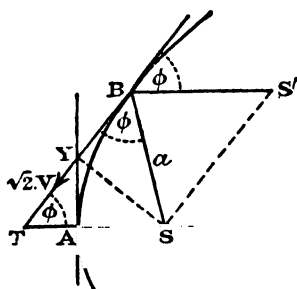


Fig. 155.

$$\therefore \quad l = 2a \frac{b^2}{a^2} \\ = \frac{2b^2}{a},$$

i.e., the latus rectum of the parabola is double that of the ellipse.

**106. Tangential and Normal Accelerations.**—If  $v$  is the velocity of a point P moving in a plane curve, the tangential acceleration is  $\frac{d^2s}{dt^2}$ , and the normal acceleration is  $\frac{v^2}{\rho}$ , where  $s$  is the distance of P from some fixed point on the curve, and  $\rho$  the radius of curvature at P.

If then T and N denote the forces per unit mass acting on a particle in the directions of the tangent and normal, the equations of motion will be

$$\ddot{s} \left( = v \frac{dv}{ds} \right) = T \quad \dots \quad (1)$$

$$\text{and} \quad \frac{v^2}{\rho} = N. \quad \dots \quad (2)$$

**Ex. VIII.** A small bead is projected with any velocity along a smooth circular wire under the action of a force varying inversely as the fifth power of the distance from a centre of force situated on the circumference. Prove that the pressure on the wire is constant.

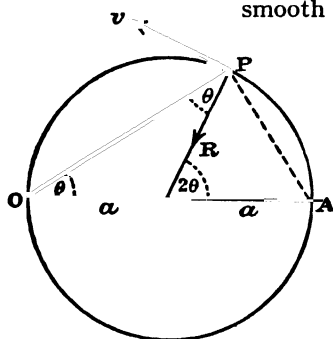


Fig. 156.

Let  $a$  be the radius, O the centre of force, A the initial position, and P the position of the particle after time  $t$ , so that  $\widehat{POA} = \theta$ .

Resolving tangentially,

$$\ddot{s} = -\frac{\mu}{r^5} \sin \theta. \quad \dots \quad (1)$$

$$\text{But} \quad s = a \cdot 2\theta, \quad r = 2a \cos \theta,$$

$$\therefore \quad 2a\ddot{\theta} = -\frac{\mu}{2^5 a^5} \cdot \frac{\sin \theta}{\cos^5 \theta}. \quad \dots \quad (2)$$

Integrating,

$$a\dot{\theta}^2 = C - \frac{\mu}{2^7 a^5} \cdot \frac{1}{\cos^4 \theta}. \quad \dots \quad (3)$$

$$\text{Let} \quad \dot{\theta} = \omega, \quad \text{when} \quad \theta = 0$$

$$\text{so that} \quad C = a\omega^2 + \frac{\mu}{2^7 a^5}$$

$$\text{and} \quad \therefore \quad a\dot{\theta}^2 = a\omega^2 + \frac{\mu}{2^7 a^5} \left( 1 - \frac{1}{\cos^4 \theta} \right). \quad \dots \quad (5)$$

Resolving normally

$$\frac{ms^2}{a} = R - \frac{m\mu}{r^5} \cos \theta.$$

But  $\dot{s} = 2a\dot{\theta}$  and  $r = 2a \cos \theta$

$$\begin{aligned} \therefore R &= m \left\{ 4a\omega^2 + \frac{\mu}{2^5 a^5} \left( 1 - \frac{1}{\cos^4 \theta} \right) + \frac{\mu}{2^5 a^5 \cos^4 \theta} \right\} \\ &= 4m \left( a\omega^2 + \frac{\mu}{2^7 a^5} \right), \end{aligned}$$

and is therefore constant throughout the motion.

**107. Motion of a Heavy Particle on a Rough Curve in a Vertical Plane.**—Let  $\phi$  be the angle which the tangent makes with the horizontal, and  $\mu$  the coefficient of friction.

Suppose also that  $\phi$  increases as  $s$  increases and that  $R$ , the pressure, is measured outwards.

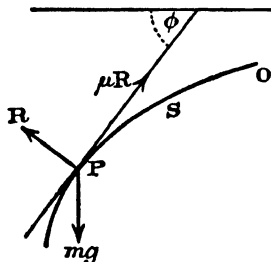


Fig. 157.

Resolving tangentially

$$mv \frac{dv}{ds} = mg \sin \phi - \mu R. \quad (1)$$

Resolving along the normal

$$m \frac{v^2}{\rho} = mg \cos \phi - R. \quad (2)$$

Eliminating  $R$  by multiplying (2) by  $\mu$  and subtracting, we have

$$v \frac{dv}{ds} - \mu \frac{v^2}{\rho} = g(\sin \phi - \mu \cos \phi). \quad \therefore (3)$$

But since  $\rho = \frac{ds}{d\phi}$ , equation (3) becomes

$$\frac{d(v^2)}{d\phi} - 2\mu v^2 = 2g\rho(\sin \phi - \mu \cos \phi): \quad \therefore (4)$$

Equation (4) can be integrated when  $\rho$  is known in terms of  $\phi$ .

Thus, if the particle is sliding down a portion of a cycloid, vertex upwards,

$$s = 4a \sin \phi$$

$$\therefore \rho = \frac{ds}{d\phi} = 4a \cos \phi,$$

so that (4) becomes

$$\begin{aligned} \frac{d(v^2)}{d\phi} - 2\mu v^2 &= 4ag(\sin 2\phi - 2\mu \cos^2 \phi) \\ &= 4ag(\sin 2\phi - \mu - \mu \cos 2\phi). \end{aligned} \quad (5)$$

To integrate this equation multiply by  $e^{-\mu\phi}$ , and we have

$$v^2 e^{-2\mu\phi} = 4ag \int e^{-2\mu\phi} (\sin 2\phi - \mu - \mu \cos 2\phi) d\phi + C. \quad (6)$$

$$\text{Now } \int e^{-2\mu\phi} \sin 2\phi d\phi = -\frac{e^{-2\mu\phi}}{2(\mu^2 + 1)} (\mu \sin 2\phi + \cos 2\phi)$$

$$\text{and } \int e^{-2\mu\phi} \cos 2\phi d\phi = \frac{e^{-2\mu\phi}}{2(\mu^2 + 1)} (\sin 2\phi - \mu \cos 2\phi)$$

$$\therefore v^2 e^{-2\mu\phi} = 2age^{-2\mu\phi}$$

$$-\frac{2ag}{1 + \mu^2} e^{-2\mu\phi} \left\{ 2\mu \sin 2\phi - (\mu^2 - 1) \cos 2\phi \right\} + C: \quad (7)$$

The equation cannot be integrated further.

#### Problems XIV.

1. A particle of unit mass falls from rest under gravity in a medium whose resistance is  $k$  times the velocity; if the particle is projected initially horizontally with a velocity  $u$ , find the horizontal and vertical velocities after a time  $t$ , and hence prove that the hodograph of its path is a straight line.

2. A particle moves in a semicircle under the action of a force perpendicular to the bounding diameter; show that the force varies inversely as the cube of the distance of the particle from the diameter.

3. Find the path of a particle acted on by a repulsive force always perpendicular to a given straight line and proportional to the distance from it, the velocity at any point being that which would be acquired by moving from rest on the given line to that point.

4. If a particle be acted upon by a force always parallel to the axis of  $y$  and proportional to the square of the radius of curvature at the point, prove that it will describe the curve

$$\frac{y - b}{a} = \log \sec \frac{x}{a},$$

the particle moving parallel to the axis of  $x$  at the point  $(0, b)$ .

5. A particle is projected under gravity, and the horizontal component of the velocity in the trajectory is proportional to the cotangent of the inclination of the direction of motion to the horizontal. Show that the resistance of the air (assumed a function of the velocity) is proportional to the velocity.

6. A particle is projected horizontally with velocity  $u$ , and it moves under the action of a vertical force proportional to  $\frac{1}{(a - y)^2}$ , where  $a$  is a constant. Taking the horizontal and vertical directions through the point of projection as the axes of  $x$  and  $y$  respectively, find the equation of the path.

7. A particle moves in a plane under a constant acceleration  $\mu a$  parallel to  $Ox$ , and an acceleration  $-2\mu y$  parallel to  $Oy$ , where  $Ox, Oy$  are rectangular axes. If the particle starts from rest at the point  $(0, a)$  show that, when  $y$  is first equal to  $\frac{a}{2}$ ,  $x$  is equal to  $\frac{\pi^2}{36}a$ .

8. A particle moves in a plane under the action of a force to a fixed point in the plane, attracting directly as the distance from the point. Prove that the path is an ellipse, with centre at the fixed point, and that the velocity at any point varies as the diameter conjugate to that point.

9. A particle of mass  $m$  describes an ellipse under the action of a force  $m\mu/r^2$  towards the focus. Prove that the average value of the kinetic energy taken with respect to the time over a complete revolution is  $m\mu/2a$ , where  $2a$  is the major axis of the ellipse.

10. A particle moves in an orbit under a central acceleration  $\mu r^{-2}$  along the radius vector; obtain the equations of energy and angular momentum. The particle is projected with velocity  $u$  at right angles to the radius at distance  $c$  from the origin; prove that  $c$  is its least distance from the origin if  $u^2 > \mu/c$ .

11. A particle describes an equiangular spiral under the action of a force to the pole. Prove that the force varies inversely as the cube of the distance of the particle from the pole.

12. A particle is describing a circle under the action of a force to a point on its circumference. Prove that the law of force varies inversely as the fifth power of the distance.

13. A body is describing a circle in a central orbit, the force varying inversely as the square of the distance. If the attraction suddenly decreases by one-half, show that the subsequent path is a parabola.

14. Prove that the law of central force in a conic about the focus is  $n^2a^3/r^2$ , where  $2a$  is the major axis, and  $2\pi/n$  is the period.

Calculate to the nearest minute the sun's greatest angular motion in one day, given that the eccentricity of the orbit is  $\frac{1}{60}$ .

15. A particle is describing an ellipse under an attraction to the focus. When it is at any point of the path the centre of attraction is transferred to the other focus, and the attraction continues to vary as the inverse square of the distance, but the absolute force is changed so that the magnitude of the attraction is unaltered; show that the particle continues to describe the same ellipse.

16. A particle is describing a hyperbola about a centre of force attracting as the inverse square of the distance, its velocity at infinity being  $V$  along an asymptote whose perpendicular distance from the centre of force is  $p$ . Find the angle between the asymptotes of the orbit, and show that, when  $V$  is large, this is approximately

$$\frac{2\mu}{pV^2}.$$



17. A particle moves with a central acceleration  $\mu\left(r + \frac{a^4}{r^3}\right)$

being projected from an apse at a distance  $a$  with a velocity  $2a\sqrt{\mu}$ ; show that it describes the curve  $r^2(2 + \cos \sqrt{3}\theta) = 3a^2$ .

18. Two particles of masses  $m_1, m_2$  attract one another according to the law of force  $\frac{\gamma m_1 m_2}{r^2}$ : prove that, if the relative orbit is a circle, the periodic time is equal to

$$2\pi \sqrt{\frac{r^3}{\gamma(m_1 + m_2)}}$$

In the case of the earth and moon, taking their distance apart to be 60 times the earth's radius, the length of the seconds pendulum to be 39 inches, the radius of the earth to be 3,960 miles, and the mass of the moon to be  $\frac{1}{80}$  of the mass of the earth, express the period in days.

19. In a central orbit the force is  $\mu u^3(3 + 2a^2 u^2)$ ; if the particle be projected at a distance  $a$  with a velocity  $\sqrt{\frac{5\mu}{a^2}}$  in a direction making  $\tan^{-1} \frac{1}{2}$  with the radius, show that the equation to the path is  $r = a \tan \theta$ .

20. A particle moves under a central force  $P$  in a medium whose resistance is  $k$  times the velocity. Prove that  $\frac{d^2 u}{d\theta^2} + u = \frac{P}{h^2 u^2} e^{2ku}$  where  $(u)$  is the reciprocal of the radius vector, and  $(b)$  a constant.

Hence or otherwise, show that when the force (attractive) varies as the distance, and the resistance is small, the areas swept over by the radius vector in successive revolutions of the particle, diminish according to the law  $e^{-Mu}$ .

21. A particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the law of force is that of the inverse cube, and that the path is an equiangular spiral.

22. A particle is acted on by an attractive force  $\mu r^{-n}$  ( $n < 3$ ) to a fixed point and its orbit is nearly circular; prove that the apsidal angle is equal to  $\pi/(3 - n)^{\frac{1}{2}}$ .

23. Prove that when the orbit under a central force is a conic with the centre of force at the focus, if  $P$  is the particle,  $S$  the centre of force and  $N$  the foot of the perpendicular from  $P$  on the axis of the conic, the velocity of  $N$  is greatest when it coincides with  $S$ .

24. A particle  $P$  describes a parabola under the action of a force tending to the focus; determine the law of force and prove that, if  $M$  is the foot of the perpendicular from  $P$  on the directrix, the motion of  $M$  is the same as that of a particle constrained to slide without friction along the directrix and attracted to the focus by a force proportional to the inverse fifth power of the distance from the focus.

25. Prove that the time required to describe an arc of a parabola under the action of the force  $\mu u^2$  to the focus, starting from the nearer

end of the axis is given by

$$\sqrt{\frac{2a^3}{\mu}} \left\{ \tan \frac{\theta}{2} + \frac{1}{3} \tan^3 \frac{\theta}{2} \right\}$$

where  $a$  is the apsidal distance, and  $\theta$  is measured from the axis.

26. A heavy bead slides down a rough circular wire in a vertical plane. If the coefficient of friction is  $\cdot 25$ , and if the particle starts from rest at the end of a horizontal diameter, show that when it has described an angle of  $\frac{\pi}{3}$  radians about the centre, the velocity  $v$

is given by  $v^2 = \frac{g}{10a} \left[ 7\sqrt{3} + 6 - 12e^{-\frac{\pi}{6}} \right]$ , where  $a$  is the radius of the wire.

27. A bead slides down a rough circular wire, coefficient of friction  $\frac{1}{\sqrt{2}}$ , whose plane is vertical, starting from rest at the end of a horizontal diameter. Prove that the bead will come to rest before reaching the lowest point.

28. Show that the only law for a central attraction, for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to that distance, is that of the inverse cube.

29. A particle is projected from the lowest point along the inside of a smooth elliptic ring, the minor axis of the ellipse being vertical. By finding the pressure between the particle and the ring in any position, show that the least velocity of projection that will carry the particle completely round the ring without losing contact is given by  $u^2 = g(a^2 + 4b^2)/b$ , where  $a$ ,  $b$  are the semi-axes.

30. A particle, attached to a fixed point O by an inelastic string of length  $r$ , is let fall from a point in the horizontal through O at a distance  $r \cos \theta$  from O. Show that the velocity of the particle, when it is vertically below O, is  $\sqrt{2gr(1 - \sin^2 \theta)}$ .

31. A rough hemispherical bowl radius  $a$ , coefficient of friction  $\mu$ , is fixed with its rim in a horizontal plane. A particle falls from rest at a point on the rim. What is its height from the lowest point of the bowl when next it comes to rest?

32. A particle is describing an ellipse of eccentricity  $e$  about a centre of force attracting as the inverse square of the distance and when at the apse farther from the centre of force its speed is suddenly doubled with the direction of motion unchanged. Prove that if  $e < \frac{1}{2}$ , the new orbit will be a hyperbola of eccentricity  $3-4e$ .

33. A particle describes the ellipse  $b^2x^2 + a^2y^2 = a^2b^2$  in such a manner that the eccentric angle of its position on the ellipse increases at a uniform time-rate  $\omega$ . Show that its acceleration is directed towards the centre of the ellipse and that the magnitude of this acceleration is  $r\omega^2$ , where  $r$  is the distance from the centre; show that the magnitude of this velocity is  $b'\omega$ , where  $b'$  is the semi-diameter of the ellipse parallel to the tangent at the point occupied by the particle.

## CHAPTER XV

### SHEARING STRESSES, BENDING MOMENTS AND ELASTICITY

**108. Theory of the Distribution of Shearing Force and Bending Moment in a Horizontal Beam subject to Vertical Loads.**—In what follows we shall assume that the vertical loads acting on the beam are in one plane, which will be taken as the plane of the paper.

Let AB be a beam in equilibrium under the action of any number of vertical forces,  $P_1, P_2, P_3$ , etc. Consider any portion CB of the beam, where C is a vertical section at any

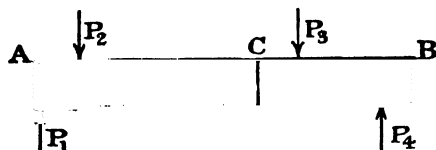


Fig. 158.

point in the beam, and B one end. Since CB is in equilibrium, the action of the portion AC on CB must balance the forces acting on CB. But those forces are equivalent to a

vertical force  $S$  acting at C and a couple  $M$ . Hence the action of AC on CB must also be a force  $S$  perpendicular to the beam and a couple  $M$ .

The force  $S$ , which tends to cause adjacent vertical sections of the beam to slide over one another is called the **shearing force** or **shear**, and the moment  $M$ , the **bending moment**, or **moment of flexure**, at C.

Thus the shear is the algebraical sum of all the vertical forces on one side of C, and the bending moment the algebraical sum of the moments of the forces on one side of C about a horizontal axis in that section.

**NOTE 1.** Any beam used in construction must therefore be strong enough to resist both the bending moment and the shearing action at any point. These actions are independent of one another, and in most cases the material necessary to resist the bending moment is much more than sufficient to resist the shearing action, so that the latter is left out of account when determining the required dimensions of the beam.

Since shearing force and bending moment are mutual actions, their directions on the two portions of the beam at C will be equal and opposite. We shall assume as the positive senses of this force and couple those shown in figure 159.

**109. Isolated Loads** (weight of beam neglected).—Resolving vertically for the external forces on CB, we obtain

$$S = P_3 - P_4 \quad \dots (1)$$

But this equation will remain true, as C approaches  $P_3$ , i.e., the

shear will remain constant over any portion of the beam which has no isolated load acting in it.

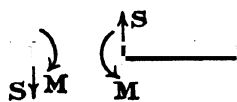


Fig. 159.

Between  $P_3$  and  $P_4$ ,  $S = -P_4$ , and therefore at the point of application of an isolated load  $P_3$  there is a sudden decrease in the shear by an amount  $P_3$ , and then the shear again remains constant till the point of application of  $P_4$  is reached.

Let  $P_4$  act at distance  $y$  and  $P_3$  at distance  $x$  from C.

Then, taking moments about C for the equilibrium of CB,

$$M = P_3x - P_4y. \quad \dots \dots (2)$$

If, again, we imagine C to approach  $P_3$ , then  $x$  decreases continuously as M approaches and finally equals the value  $P_4y$ . Hence, there is no sudden alteration in the bending moment on passing the point of application of an isolated load.

### 110. Bending Moment and Shearing Force Diagrams.—

If at every point C along the beam we erect two perpendiculars which are proportional to the shearing force and bending moment respectively at that point, diagrams are obtained which are called the shearing force and bending moment diagrams. It is necessary to distinguish in the usual way between the positive and negative values of the quantities considered.

From a consideration of the preceding paragraph we see that, for any system of isolated forces, the shearing force diagram will consist of a series of isolated horizontal lines, and the bending moment diagram of a continuous series of sloping lines.

**Ex. I.** AB is a cantilever of length 10 feet, with load 2 tons at the free end B, and load 4 tons at D, 4 feet from the free end and load 5 tons at C, 3 feet from A. Construct the shearing force and bending moment diagrams.

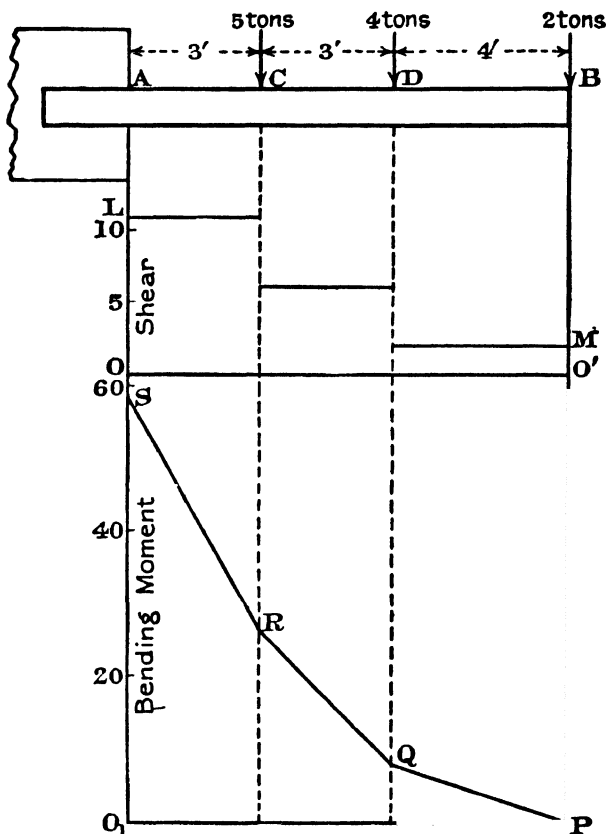


Fig. 160.

The shear between A and C is 11 tons  
 C and D is 6 tons  
 D and B is 2 tons } all *positive*.

The ordinates (from  $OO'$ ) of the broken line LM represent these shearing forces.

The bending moment from B to D increases continuously from 0 to  $2 \times 4 (= 8)$  therefore its graph is the straight line PQ. From D to C the bending moment increases continuously from 8 to 26 and we thus get the straight line QR. In similar manner we get RS, and thus the bending moment curve is PQRS.

**111. Continuous Loads.**—Let a horizontal line through the beam be taken as the axis of  $x$ , and let the co-ordinate of any point C, measured to the right be  $x$ . Let  $w$  be the load per unit length, so that the load on an element  $\delta x$  is  $w\delta x$ .

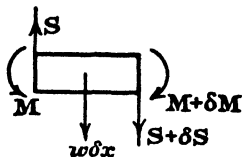


Fig 161.

Consider the equilibrium of the element  $\delta x$ .

The equations (1) and (2) of art. 109 then become

$$\delta S + w\delta x = 0 \quad \dots \quad (3)$$

$$\text{and} \quad \delta M + (S + \delta S)\delta x + w\delta x \cdot e\delta x = 0 \quad \dots \quad (4)$$

where  $e$  is some proper fraction.

When  $\delta x$  becomes indefinitely small, equations (3) and (4) become

$$\frac{dS}{dx} = -w \quad \dots \quad (5)$$

$$\text{and} \quad \frac{dM}{dx} = -S. \quad \dots \quad (6)$$

In the case of a uniform load,  $w$  is constant and we then obtain by integration from equations (5) and (6)

$$S = -wx + C \quad \dots \quad (7)$$

$$\text{and} \quad M = -\int S dx = +\int (wx - C) dx \\ = \frac{1}{2}wx^2 - Cx + D. \quad \dots \quad (8)$$

The constants  $C$  and  $D$  can be determined from the conditions of the given problem. Thus at a free end  $M$  is zero.

Equation (7) shows that the graph of the shearing force is a straight line, and equation (8) shows that the graph of the bending moment is a parabola with its axis vertical.

From equation (6) we see that the shearing force at any section C is a measure of the slope of the bending moment graph at the point corresponding to C.

$$\text{Again, since} \quad \int dM = -\int S dx$$

$$\therefore \quad M_1 - M_2 = \int S dx.$$

i.e., the difference in the moments at any two sections is determined by the area of the shearing force diagram between the points corresponding to the two sections.

**Ex. II.** A uniform beam of weight  $W$  tons and length  $l$  is supported at its ends A and B in a horizontal position and carries two loads  $P$  tons at C and  $Q$  tons at D at distances  $x_1$  and  $x_2$  from A respectively. Draw the shearing force and bending moment diagrams.

The reactions  $R_1$  and  $R_2$  may be determined by taking moments about B and A respectively.

The shearing force at B is  $-R_2$ .

From B to D there is, in addition, a positive shearing force due to the weight of a portion of the rod. At D there is discontinuity in the diagram due to the addition of the force  $Q$  tons. The complete shearing force diagram is shown in the figure.

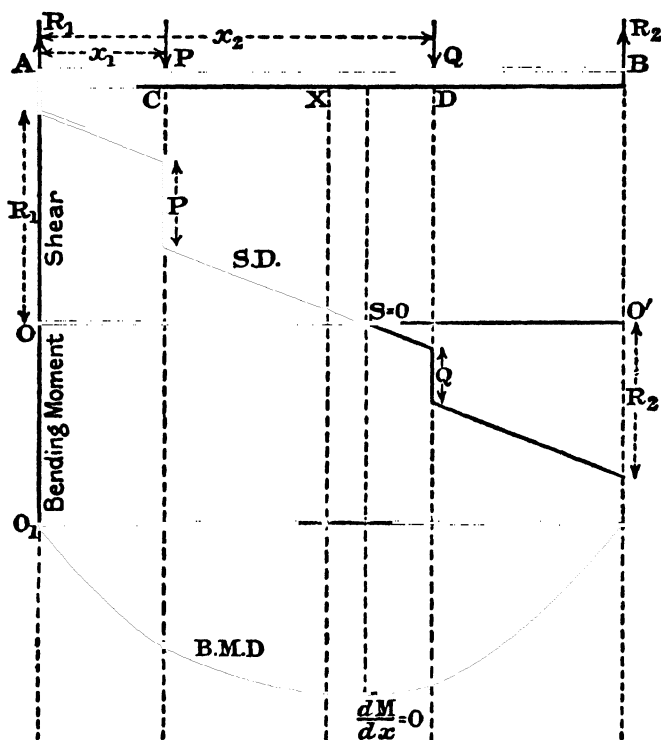


Fig. 162.

The bending moment at any point X will consist of the moments of the isolated loads on the right of X together with the moment due to the weight of the portion XB.

Between D and B the bending moment, at a distance  $x$  from A is given by

$\frac{W}{l} \frac{(l-x)^2}{2} - R_2(l-x)$ ; since  $\frac{W}{l}$  is the weight of the beam per unit length; ... (1)

between C and D, it is

$$\frac{W}{l} \frac{(l-x)^2}{2} - R_2(l-x) + Q(x_2 - x) \quad \dots \quad (2)$$

and between A and C, it is

$$\frac{W}{l} \frac{(l-x)^2}{2} - R_2(l-x) + Q(x_2-x) + P(x_1-x) \quad (3)$$

or otherwise  $R_1x - \frac{W}{l} \cdot \frac{x^2}{2}$  by considering the forces to the left of C.

Hence, in each case the curves will be portions of parabolas with axes vertical. Moreover since the coefficient of  $x^2$  in all three cases is the same, the magnitude of the latus rectum of each parabola will be the same. Each curve will intersect the next on the vertical line through the point at which an isolated load acts. Equation (6) of the previous paragraph is still true; so that the slope of the bending moment diagram gives the shear.

Thus in (1) above

$$\frac{dM}{dx} = -\frac{W}{l}(l-x) + R_2 = -S$$

$$\therefore S = \frac{W}{l}(l-x) - R_2$$

and this is seen to be true by examination of the shearing force diagram. At the point where  $S = 0$ , the bending moment is numerically a maximum.

**Ex. III.** A horizontal beam is supported at one end and at one other point by a prop in the same horizontal line. If the beam is loaded with a uniformly distributed load, find where the prop must be placed so that the maximum bending moment on the beam is as small as possible.

Let ABC be the beam, supported at A and B.

Let  $AC = 2l$  and  $AB = y$  and let  $R_1$  and  $R_2$  be the reactions at A and B.

If  $w$  is the load per unit length, then  $R_1 = 2wl\left(1 - \frac{l}{y}\right)$  and  $R_2 = \frac{2wl^2}{y}$ .

If  $x$  is the distance of a point P, between B and C, from C, then the bending moment at P  $= \frac{wx^2}{2}$ .

This bending moment is evidently greatest at B, and is then equal to  $\frac{w}{2}(2l-y)^2$ .

Next, considering a point between A and B, distant  $x$  from A, we obtain for the bending moment there

$$R_1x - \frac{wx^2}{2}.$$

For a maximum bending moment  $x = \frac{R_1}{w}$  and therefore the maximum bending moment is



$$\frac{R_1^2}{w} - \frac{R_1^2}{2w} = \frac{R_1^2}{2w} = 2wl^2 \left(1 - \frac{l}{y}\right)^2.$$

One of these bending moments will in general, be the greater but, by moving the prop, we can decrease it.

Hence, we shall obtain the smallest value of the maximum bending moment when the position of the prop is such that the above bending moments are equal ; i.e. when

$$2wl^2 \left(1 - \frac{l}{y}\right)^2 = \frac{w}{2} (2l - y)^2$$

i.e. when 
$$2l \left(1 - \frac{l}{y}\right) = \pm (2l - y)$$

which gives

$$y = \pm \sqrt{2}l,$$

or

$$y = (2 + \sqrt{2})l.$$

Since  $y$  is necessarily positive, and less than  $2l$ , and from consideration of the equilibrium of the beam, greater than  $l$ , the only possible solution is  $y = \sqrt{2}l$ .

### 112. Elasticity.

In Chapter II, the case of the elastic string, where a knowledge of Hooke's Law was required, has already been discussed. This law also applies to the compression and elongation of elastic rods, and is the basis of the theory of elastic solids. The law only applies within certain limits, called the limits of elasticity, and beyond those limits, the rod or other solid suffers permanent deformation.

Consider a rod AB of original length  $l$ , in a state of tension due to forces  $W, W$  acting in opposite directions at its ends. The bar has become longer and what is called a **tensile strain** has been produced.

If  $x$  is the increase in length, then the tensile strain  $= \frac{x}{l}$ .

Corresponding to the external forces and balancing them there is a set of equal and opposite internal forces at any section C of the bar. This system of internal forces is called the **tensile stress**.

If  $a$  be the area of cross section at C, and  $f$  denote the intensity of stress at the section, then

$$W = af. \quad \dots \dots \dots (1)$$

Now Hooke's Law states that the strain produced is directly proportional to the load, which is itself proportional to the stress by (1), i.e. the ratio  $\frac{\text{intensity of stress}}{\text{strain per unit length}}$  is constant up to the elastic limit.

This ratio is called the modulus or coefficient of elasticity and is denoted by  $E$ .

Hence in the case of the rod considered,

$$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{x/l} = \frac{fl}{x} \quad \dots \quad (2)$$

whence 
$$f = \frac{Ex}{l} \quad \text{or} \quad x = \frac{fl}{E}.$$

For steel,  $E$  is about 30,000,000 lbs. per sq. inch.

„ wrought iron „ 28,000,000 „ „ „ „

„ cast iron „ 15,000,000 „ „ „ „

„ oak „ 1,500,000 „ „ „ „

„ leather „ 25,000 „ „ „ „

NOTE 2. When a rod is stretched, its breadth and thickness are also altered. These changes also follow a law similar to Hooke's Law.

**113. Stress-Strain Diagrams.**—If a bar be stretched by an increasing load, and the strains and corresponding stresses plotted we get a graph which is a straight line from  $O$  to  $X$ , where  $X$  represents the elastic limit. Beyond that limit the graph is more or less irregular.

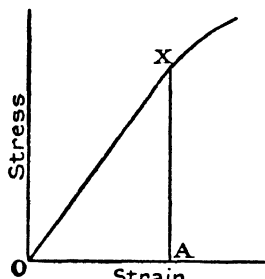


Fig. 163.

*Work done during extension.*—If a rod be stretched a distance  $b$ , and if  $W$  be the load then acting, the work done during the extension is  $\frac{1}{2}Wb$ ; for the force acting varies directly as the extension and increases uniformly from 0 to  $W$ , therefore the average force acting is  $\frac{1}{2}W$  and the work done is  $\frac{1}{2}Wb$ .

The area  $OXA$  in the diagram, if suitable units be chosen, will represent the work done.

**114. Stress due to a Sudden Load on a Bar.**—Let a weight  $W$ , threaded on a vertical rod which is suspended from its upper end, fall from a height  $h$  on a head at the lower end of the bar.

Let this blow cause an extension  $x$  in the bar.

Then work done on the bar = energy given up by the falling weight, if any loss due to generation of heat be neglected

$$= W(h + x).$$

Let  $f$  be the maximum stress produced,  $a$  the area of cross-section and  $l$  the length of the rod.

Then 
$$W(h + x) = \frac{afx}{2}$$

But 
$$x = \frac{fl}{E}$$

$$\therefore W\left(h + \frac{fl}{E}\right) = \frac{af^2l}{2E}.$$

Now, suppose  $h = 0$ , i.e. suppose a load  $W$  is suddenly put on the rod, then  $f = \frac{2W}{a}$ .

If, on the other hand, the load is put on gradually, then finally  $W = af$ , or  $f = \frac{W}{a}$ .

Hence the effect of a suddenly applied load is double that produced when the same load is applied gradually.

**115. Kinds of Stress and Strain.**—There are three kinds of stress and strain :—

- (1) Tensile strain and tensile stress ;
- (2) Compressive strain and compressive stress ;
- (3) Shearing strain and shearing stress.

The discussion of the second case is more complex than that of the first, for a bar subjected to compression has a tendency to bend, and the compressive stress at any transverse section is then not uniform.

**116. Shearing Strain and Shearing Stress.**—The modulus of elasticity in any case is the ratio of the intensities of stress and strain.

In the case of shearing stress, the strain is measured by the distortion produced.

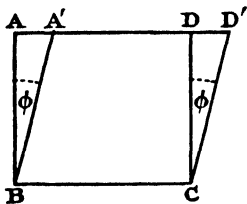


Fig. 164.

Thus, suppose a block ABCD, fixed at BC, is acted on by a tangential or shearing force  $P$  acting along AD and producing a distortion of angle  $\phi$ .

Then the shearing strain produced is measured by  $\tan \phi$  or  $AA_1/AB$ . If  $C$  is the modulus of shearing stress and  $f$  the intensity of shearing stress produced,

then 
$$C = \frac{f}{\tan \phi},$$

or, since  $\phi$  is small,  $f = C\phi$ , where  $\phi$  is measured in radians.

For steel,  $C$  is about 13,000,000 lbs. per sq. inch.

„ wrought iron „ 10,000,000 „ „ „ „  
 „ cast iron „ 6,000,000 „ „ „ „

$C$  is often called the modulus or coefficient of rigidity.

**117. Torsion of a Bar of Circular Section.**—Consider a bar of length  $l$ , radius  $R$  subjected to torsion by a couple applied at one end, the other end being fixed.

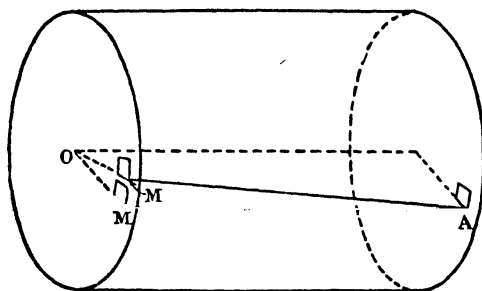


Fig. 165.

Take a thin rectangular prism of the material running parallel to the axis of the cylinder, one edge of which is  $AM$ ,  $A$  being fixed.

The edge  $AM$  will be turned into the position  $AM'$ .

Let  $\widehat{MAM'} = \delta\phi$  and let  $\widehat{M'OM} = \delta\theta$ , where  $O$  is the centre of the end section, and let  $OM = OM' = r$ .

Then the face of the square at  $MM'$  will be subjected to a shearing stress of intensity  $f$ ; where  $f = C\delta\phi$ .

$$\text{But} \quad MM' = l\delta\phi = r\delta\theta \quad \therefore \quad \delta\phi = \frac{r\delta\theta}{l}.$$

$$\text{so that} \quad f = C \cdot \frac{r}{l} \delta\theta. \quad \dots \quad \dots \quad \dots \quad (1)$$

Every portion of the area at  $M$  is twisted through the same angle, i.e.  $\delta\theta$  is constant, therefore  $C\delta\theta/l$  is constant and therefore the shear stress is proportional to  $r$ . Now consider an annulus at distance  $r$  from  $O$ . Since the shear stress is proportional to  $r$ , it is therefore constant round the annulus, and

$$\text{shear force acting round annulus} = \frac{C\delta\theta}{l} r \cdot 2\pi r \delta r.$$

$$\begin{aligned} \text{Moment of this force about the} \\ \text{axis of the cylinder} &= \frac{C\delta\theta r}{l} \cdot 2\pi r \delta r \cdot r \\ &= \frac{2\pi C\delta\theta}{l} r^3 \delta r. \end{aligned}$$

Hence, total moment of resistance of the shaft to torsion. 
$$= \frac{2\pi C \delta \theta}{l} \int_0^R r^3 dr$$

$$= \frac{\pi C}{l} \cdot \frac{R^4}{2} \cdot \delta \theta.$$

But the moment of inertia of a circle of radius  $R$  about an axis through its centre and perpendicular to its plane is  $\frac{\pi R^4}{2}$ . Hence, denoting this moment of inertia by  $I$ , and the moment of resistance by  $M$ , we have

$$M = \frac{CI\delta\theta}{l} \quad \dots \quad \dots \quad (2)$$

But, from equation (1)  $\frac{C\delta\theta}{l} = \frac{f}{r}$ , so that if  $F$  is the maximum stress which is evidently exerted at radius  $R$ ,

$$\frac{C\delta\theta}{l} = \frac{F}{R}, \quad \dots \quad \dots \quad (3)$$

from which

$$M = \frac{F}{R} \cdot I. \quad \dots \quad \dots \quad (4)$$

Again

$$I = \frac{\pi R^4}{2}$$

$$\therefore M = \frac{F}{R} \cdot \frac{\pi R^4}{2} = \frac{\pi R^3}{2} F. \quad \dots \quad (5)$$

NOTE 3. In the case of an engine shaft the permissible stress  $F$  is known and  $M$  is calculated from the pressure on the piston and the length of the crank. Then equation (5) gives the suitable diameter of the crank shaft.

### 118. Moment of Resistance of a Hollow Cylindrical Shaft.—

Let  $R$  and  $r$  be the inner and outer radii of the shaft.

The moment of resistance of a solid of radius  $R$  is

$$\frac{\pi R^3}{2} F,$$

where  $F$  is the shear force at radius  $R$ .

The moment of resistance of a solid shaft of radius  $r$ , if it forms the central portion of the other is

$$\frac{\pi r^3}{2} f$$

where  $f$  is the shear stress at radius  $r$ .

But 
$$\frac{f}{r} = \frac{F}{R} \quad \therefore f = \frac{rF}{R}.$$

Hence the moment of resistance of the hollow shaft is

$$\begin{aligned} \frac{\pi R^3}{2} F - \frac{\pi r^3}{2} \cdot \frac{rF}{R} \\ = \frac{\pi}{2R} (R^4 - r^4) F. \end{aligned}$$

**119. Torsional Pendulum.**—Suppose the upper end of a wire of length  $l$  and radius  $r$  be fixed and the lower end attached to a rigid body. Then when the body is rotated an angle  $\theta$  from its position of equilibrium about the vertical line containing the wire, the couple due to the torsion of the wire tending to restore equilibrium will, by Art. 117, be

$$\frac{C\pi r^4}{2l} \theta$$

where  $C$  is the coefficient of rigidity of the substance composing the wire. Thus if  $M$  be the mass of the rigid body and  $k$  its radius of gyration about the axis of rotation, we have

$$Mk^2 \frac{d^2\theta}{dt^2} = - \frac{C\pi r^4}{2l} \theta,$$

$C$  being measured in theoretical units of force per unit area.

$$\therefore \frac{d^2\theta}{dt^2} = - \frac{C\pi r^4}{2lMk^2} \theta.$$

Oscillations therefore take place in accordance with Simple Harmonic Motion and the periodic time is

$$2\pi \sqrt{\frac{2lMk^2}{C\pi r^4}}.$$

**Ex. IV.** From a steel wire 10 ins. long and radius .025 in. is suspended a sphere radius  $1\frac{1}{2}$  ins. and density 480 lb. per cub. ft. If the sphere be given a displacement about its vertical diameter and then let go, calculate the periodic time of the oscillations that take place.

Periodic time

$$\begin{aligned} &= 2\pi \times \sqrt{\frac{\frac{5}{3} \times \frac{4\pi}{3} \times \frac{480}{8^3} \times \frac{2}{5} \times \frac{1}{8^2}}{13 \times 10^6 \times 32 \times 144 \times \pi \times \frac{1}{480^4}}} \text{ sec.} \\ &= 2 \times 3.14159 \times \sqrt{\frac{3}{260}} \text{ sec.} \\ &= .6749 \text{ sec.} \end{aligned}$$

**120. Bending of Beams.**—A beam originally straight is acted on by two equal and opposite couples at its ends. The beam will then become curved. As the action at any section

of the beam must be such as to balance the couple at the end of the beam it is evident that all the separate elements of the beam will be similarly deformed and the whole beam must take the shape of a circular arc.

In this deformation there will be no shearing of adjacent sections relatively to one another.

If, however, a horizontal rod or beam be slightly bent by vertical forces applied to it, we know that at each section of the beam there is not only a bending moment but a shearing force, so that the state of strain is no longer of the simple character which exists in the case above. We shall assume,

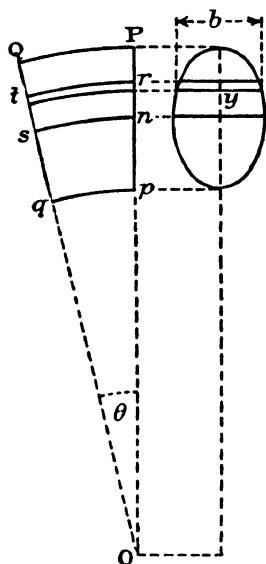


Fig. 166.

however, that the additional strains introduced are negligible, and that the results which follow apply to both cases mentioned. Suppose that after the bending has taken place, two sections,  $Pp$ ,  $Qq$ , which were originally parallel meet in a line through  $O$  perpendicular to the plane of the paper.

The portion  $pq$  is compressed and  $QP$  stretched.

Between them there will evidently be a lamina of particles meeting the paper in  $sn$ , whose length is unaltered by the bending. This lamina is called the **neutral surface** and  $sn$  the **neutral line**.

Let  $On = R$  and  $\widehat{POQ} = \theta$   
Then  $ns = R\theta$ .

If  $rt$  denote any other surface at a distance  $y$  from  $ns$ , then

$$rt = (R \pm y) \theta.$$

$$\text{Now strain of } rt = \frac{rt - ns}{ns} = \frac{y\theta}{R\theta} = \frac{y}{R}.$$

$$\text{Hence stress on } rt = f = E \times \text{intensity of strain} = \frac{Ey}{R}.$$

$$\text{i.e.,} \quad \frac{f}{y} = \frac{E}{R} \quad \dots \quad \dots \quad \dots \quad (1)$$

If  $b$  is the width of the section at  $rt$ , then the force acting on  $rt$  is

$$f \cdot b \delta y = b \delta y \frac{Ey}{R}.$$

Then the total force across the section  $Pp$  is

$$T = \frac{E}{R} \cdot \int by dy. \quad \dots \quad (2)$$

But the action at this section is, by hypothesis, a couple, and therefore  $T$  must be zero

i.e.  $\int by dy = 0. \quad \dots \quad (2)$

But if  $\bar{y}$  is the distance of the centroid of the section from  $ns$ , and  $A$  the area of the section.

$$\int by dy = A\bar{y}.$$

Hence from equation (2)  $\bar{y} = 0$ .

Therefore,  $ns$ , the neutral surface, passes through the centroid of the section.

The moment about  $ns$  of the tension on  $rt$  is

$$y \cdot \frac{E}{R} \cdot by \delta y.$$

The sum of the moments of all the tensions across the section at  $Pp$  is therefore given by

$$\begin{aligned} M &= \frac{E}{R} \int by^2 dy \\ &= \frac{E}{R} \cdot I. \quad \dots \quad (3) \end{aligned}$$

where  $I$  is the moment of inertia of the area of the section about the line through the centroid of the section and perpendicular to the plane of the paper.

Hence from (1) and (3)

$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}. \quad \dots \quad (4)$$

**121. Deflection of a Beam due to Bending.**—The last equation enables us to find the form of a rod which is only slightly flexible, for it gives us a connection between the bending moment and  $R$ .

We know that  $\frac{1}{R} = \frac{\pm \frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$ .



If we take the neutral line before bending takes place as the  $x$  axis and measure  $y$  vertically downwards from this line, then, if the beam is only slightly flexible,  $\frac{dy}{dx}$  will be small, and

$$\frac{1}{R} = \pm \frac{d^2y}{dx^2}.$$

Hence, from Art. 120

$$M = \pm EI \frac{d^2y}{dx^2}. \quad \dots \quad (1)$$

Now the bending moment is considered positive when the beam is convex upwards, and then  $\frac{dy}{dx}$  increases with  $x$ , since  $y$  is measured downwards; that is,  $\frac{d^2y}{dx^2}$  is positive and therefore the positive sign must be taken in equation (1).

We have in addition to this equation the two others of Art. 111.

$$\frac{dS}{dx} = -w \quad \dots \quad (2)$$

$$\frac{dM}{dx} = -S. \quad \dots \quad (3)$$

Differentiating (1) and substituting in (2) and (3) we have

$$S = - \frac{d}{dx} \left( EI \frac{d^2y}{dx^2} \right) \quad \dots \quad (4)$$

and 
$$w = \frac{d^2}{dx^2} \left( EI \frac{d^2y}{dx^2} \right). \quad \dots \quad (5)$$

In the case of a uniform beam  $I$  will be constant and the last two equations will then take the simpler form

$$S = - EI \cdot \frac{d^3y}{dx^3} \quad \dots \quad (6)$$

$$w = EI \frac{d^4y}{dx^4}. \quad \dots \quad (7)$$

The integral of (5) or (7) will give the equation of the curve which the rod assumes.

There will be four constants in the integral of (5) or (7). These constants can be determined from the conditions given for the two ends of the beam. Each end will give two conditions, and so there will be sufficient conditions to determine the constants. If, however, there is a load or point of

support at any other point of the beam, there will be discontinuity in the value of  $S$  and therefore of  $\frac{d^3y}{dx^3}$ , and then the portions of the beam on either side of the point of application of such a force would have to be considered separately, giving a different set of constants for the two integrals.

The conditions at the ends of the beam are as follows :—

- (i.) At a **free end** we have  $M = 0, S = 0, \therefore \frac{d^2y}{dx} = 0, \frac{d^3y}{dx^3} = 0$ .
- (ii.) At a **supported end** we have  $M = 0$  and therefore  $\frac{d^2y}{dx^2} = 0$ ; whilst  $y$  is known.
- (iii.) At a **clamped end** the value of  $M$  cannot be directly found by consideration of the external forces.  $y$  however is known and  $\frac{dy}{dx}$  is generally zero.

If the value of  $M$  can be found as a function of  $x$ , and this is always possible when the reactions at fixed supports are determinable, we can then commence by integrating (1). In other cases, notably when the rod is clamped at one or more points, we commence by integrating (7).

**Ex. V.** A uniform beam of length  $l$ , of weight  $w$  per unit length is supported at its ends. Determine its form and the deflection of the beam at the centre.

Taking the origin at one end, we have

$$EI \cdot \frac{d^2y}{dx^2} = M = -\frac{1}{2}wx(l-x).$$

Hence,

$$EI \frac{dy}{dx} = \frac{1}{6}wx^3 - \frac{1}{4}wlx^2 + C.$$

But  $\frac{dy}{dx} = 0$  when  $x = \frac{l}{2}$ , from the symmetry of the figure and therefore  $C = \frac{1}{24}wl^3$ .

Integrating again,

$$EIy = \frac{1}{24}wx^4 - \frac{1}{12}wlx^3 + \frac{1}{24}wl^3x,$$

the constant being zero since  $y = 0$  when  $x = 0$ .

This is the form which the beam assumes.

To find the sag in the middle, put  $x = \frac{l}{2}$ . The value then found is

$$\frac{5}{384} \frac{wl^4}{EI} = \frac{5Wl^3}{384EI} \text{ if } W = lw, \text{ the total load.}$$

**Ex. VI.** A uniform beam is clamped horizontally at both ends and is under a uniform load. Determine the form the beam assumes and find the deflection at the centre.

Take the origin at one end, and let  $w$  be the weight per unit length of the beam.

In this case we cannot begin by forming the expression for the bending moment, and we therefore take the differential equation

$$EI \frac{d^4 y}{dx^4} = w. \quad \dots \dots \dots (1)$$

Integrating, we get

$$EI \frac{d^3 y}{dx^3} = wx + A \quad \dots \dots \dots (2)$$

$$EI \frac{d^2 y}{dx^2} = \frac{wx^2}{2} + Ax + B \quad \dots \dots \dots (3)$$

$$EI \frac{dy}{dx} = \frac{wx^3}{6} + \frac{Ax^2}{2} + Bx + C. \quad \dots \dots \dots (4)$$

Since the beam is fixed horizontally at its ends,

$$\frac{dy}{dx} = 0 \text{ when } x = 0 \text{ and when } x = l.$$

Substituting in (4) we obtain  $C = 0$

and 
$$\frac{wl^3}{6} + \frac{Al^2}{2} + Bl = 0$$

$\therefore B = -\frac{Al}{2} - \frac{wl^2}{6}.$

Hence

$$EI \frac{dy}{dx} = \frac{wx^3}{6} + \frac{Ax^2}{2} - \frac{Alx}{2} - \frac{wl^2x}{6}.$$

Integrating,

$$EIy = \frac{wx^4}{24} + A\left(\frac{x^3}{6} - \frac{lx^2}{4}\right) - \frac{wl^2x^2}{12} + D.$$

But  $y = 0$  when  $x = 0 \therefore D = 0$

and  $y = 0$  when  $x = l$

$\therefore \frac{wl^4}{24} + Al^3\left(\frac{1}{6} - \frac{1}{4}\right) - \frac{wl^4}{12} = 0$

so that  $A = -\frac{wl}{2},$

and  $EIy = \frac{wx^2}{24} (l - x)^2.$

This equation gives the form the beam assumes.

The symmetry of the conditions shows that the maximum deflection is at the middle of the beam where  $x = \frac{l}{2}$ , and its value is

$$\frac{1}{EI} \cdot \frac{wl^4}{384}.$$

For a beam fixed at one end and supported at the other, the same method will give

$$y = \frac{wx^3}{48EI} \cdot (2x^2 - 5lx + 3l^2) \quad \dots \quad (1)$$

using the conditions (i)  $y = 0$  when  $x = 0$ ; (ii.)  $y = 0$  when  $x = l$ ;

(iii.)  $\frac{dy}{dx} = 0$  when  $x = 0$ ; (iv.)  $\frac{d^2y}{dx^2} = 0$  when  $x = l$ .

To find where the maximum deflection is, we differentiate (1).

$$\therefore \frac{dy}{dx} = \frac{w}{48EI} (8x^3 - 15lx^2 + 6l^2x). \quad \dots \quad (2)$$

Putting  $\frac{dy}{dx} = 0$ , the value of  $x$  required is obtained from the quadratic

$$8x^3 - 15lx + 6l^2 = 0$$

from which

$$x = .5785l.$$

The maximum deflection is therefore

$$\begin{aligned} \frac{wl^4}{48EI} (.5785)^2 \{2(.5785)^2 - 5 \times .5785 + 3\} \\ = \frac{.0054}{EI} \frac{wl^4}{EI}. \end{aligned}$$

In this case, there is a point of inflexion between the two ends found by differentiating (2) and equating to zero.

The value of  $x$  so found is  $\frac{1}{4}l$ .

Since  $M = EI \frac{d^2y}{dx^2}$ , the bending moment at this point is zero.

**NOTE 4.** For a uniform beam fixed horizontally at one end and free at the other, under a uniform load  $w$  per unit length the deflection at the free end is  $\frac{1}{8} \frac{wl^4}{EI}$ , and it will be found that this is three-eighths of the value obtained when the same load is concentrated at the free end.

**Ex. VII.** A beam, AB, uniformly loaded, of length  $2a$ , is supported at its two ends, and also at its middle point C, the supports all being at the same level; find the pressures on the supports.

As in the previous example, the pressures on the supports are not determinable from statical principles only, and we therefore use the equation

$$EI \frac{d^4y}{dx^4} = w. \quad \dots \quad (1)$$

Taking the origin at the end A and integrating, we have

$$EI \frac{d^3y}{dx^3} = wx + A \quad \dots \quad (2)$$

$$M = EI \frac{d^2y}{dx^2} = w \frac{x^2}{2} + Ax + B. \quad \dots \quad (3)$$

Now there is discontinuity in the value of  $S$  and therefore of  $\frac{d^3y}{dx^3}$  at  $C$ , and therefore the two portions of the beam on either side of  $C$  must be considered separately. In this case, however, on account of the symmetry of the conditions we need only obtain the form of the curve for the portion  $AC$  of the beam.

Now the bending moment is zero at  $A$  where  $x = 0$ , and therefore the constant  $B$  in (3) is zero.

Integrating again

$$EI \frac{dy}{dx} = \frac{wx^3}{6} + \frac{Ax^2}{2} + C. \quad \dots \dots \dots (4)$$

At the middle point where  $x = a$ ,  $\frac{dy}{dx} = 0$ ; substituting these values in (4) we find for  $C$  the value  $-\frac{Aa^2}{2} - \frac{wa^3}{6}$ , and (4) then becomes

$$EI \frac{dy}{dx} = \frac{w}{6}(x^3 - a^3) + \frac{A}{2}(x^2 - a^2). \quad \dots \dots (5)$$

The final integration gives

$$EIy = \frac{w}{24}(x^4 - 4a^3x) + \frac{A}{6}(x^3 - 3a^2x) \quad \dots \dots (6)$$

no constant being necessary, since  $y = 0$  when  $x = 0$ .

But  $y = 0$  when  $x = a$ , from which  $A = -\frac{3}{8}wa$ .

$$\begin{aligned} \text{Now} \quad S &= -EI \frac{d^3y}{dx^3} = -wx - A, \text{ from (2)} \\ &= \frac{3}{8}wa - wx. \end{aligned}$$

$$\text{At A, where } x = 0, \quad S = \frac{3}{8}wa$$

$$\text{At C, where } x = a, \quad S = -\frac{3}{8}wa.$$

Considering the other half of the beam, we see that the pressures on the three supports  $A, C, B$  are  $\frac{3}{16}, \frac{1}{8}$  and  $\frac{3}{16}$  respectively of the total load. Again, substituting the value of the constant  $A$  in equation (3), we obtain

$$EI \frac{d^2y}{dx^2} = \frac{wx^2}{2} - \frac{3}{8}wax = \frac{wx}{8}(4x - 3a).$$

Hence there is a point of inflexion at the point  $x = \frac{3}{4}a$ , where the bending moment is zero.

Equation (6) becomes

$$\begin{aligned} EIy &= \frac{wx}{48}(2x^3 - 3ax^2 + a^3) \\ &= \frac{wx}{24}(x - a)^2(x + \frac{1}{2}a). \quad \dots \dots \dots (7) \end{aligned}$$

**Ex. VIII.** A beam 8 ft. long and weighing 16 lb. has uniform section 3 in. wide and 2 in. deep. It is clamped horizontally at one end and the other end which is free is found to be depressed 8 in. Calculate the modulus of elasticity of the material of which the beam is composed. Also find the depression if the beam is clamped so that the width is 2 in. and the depth 3 in. and the weight which would have to be attached to the free end to make the depression 8 in.

Taking the origin at the fixed end, we have, using the notation of Art. 121,

$$EI \cdot \frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2 \quad \dots \quad \dots \quad (1)$$

$$\therefore EI \frac{dy}{dx} = -\frac{w}{6}(l-x)^3 + A \quad \dots \quad \dots \quad (2)$$

$$\therefore EI \cdot y = \frac{w}{24}(l-x)^4 + Ax + B \quad \dots \quad \dots \quad (3)$$

Now when  $x = 0$ ,  $\frac{dy}{dx} = 0$  and  $y = 0$

$$\therefore \text{from equations (2) and (3), } A = \frac{wl^3}{6} \text{ and } B = -\frac{wl^4}{24}$$

$$\therefore EIy = \frac{w}{24} \left[ (l-x)^4 + 4l^3x - l^4 \right].$$

Hence when  $x = l$  the depression is  $\frac{wl^4}{8EI}$ .

Substituting the numerical values, we have

$$\frac{2}{3} = \frac{2 \times 8^4}{8 \times \frac{6}{144} \times \frac{1}{12 \times 6^2} E}$$

$$\therefore E = 15,925,248 \text{ lbs. wt. per sq. ft.}$$

$$\text{In the second case, } I_1 = \frac{6}{144} \times \frac{1}{12 \times 4^2}.$$

Therefore if  $d$  ft. be the depression, we have

$$\frac{d}{\frac{2}{3}} = \frac{\frac{1}{I_1}}{\frac{1}{I}} = \frac{I}{I_1} = \frac{4^2}{6^2}$$

$$\therefore d = \frac{8}{27} \text{ ft.}$$

In the third case, let  $W$  lb. be the weight to be attached, then

$$EI_1 \frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2 + W(l-x) \quad \dots \quad \dots \quad (4)$$

$$\therefore EI_1 \frac{dy}{dx} = -\frac{w}{6}(l-x)^3 - \frac{W}{2}(l-x)^2 + C \quad \dots \quad (5)$$

$$\therefore EI_1 y = \frac{w}{24}(l-x)^4 + \frac{W}{6}(l-x)^3 + Cx + D. \quad (6)$$

Now when  $x = 0$ ,  $\frac{dy}{dx} = 0$  and  $y = 0$

$\therefore$  from equations (5) and (6)  $C = \frac{wl^3}{6} + \frac{Wl^3}{2}$

and  $D = -\frac{wl^4}{24} - \frac{Wl^3}{6}$

$\therefore EI_1 \cdot y = \frac{w}{24}(l-x)^4 + \frac{W}{6}(l-x)^3 + \frac{l^2}{6}(wl + 3W)x - \frac{wl^4}{24} - \frac{Wl^3}{6}$

Hence when  $x = l$ , the depression is  $\frac{l^3}{24EI_1} (3wl + 8W)$ .

Substituting the numerical values we have

$$\frac{2}{3} = \frac{64}{3EI_1} (48 + 8W)$$

but  $\frac{8}{27} = \frac{64}{3EI_1} \times 48$

$\therefore \frac{2}{3} = \frac{48 + 8W}{27 \times 6}$

$\therefore W = 7\frac{1}{2} \text{ lbs.}$

NOTE 5. The value of  $E$  obtained is a number of lb. wt. per sq. ft. for  $w$  is measured in lb. per ft.

### Problems XV.

1. A uniform horizontal beam is fixed at one end and is acted on by a vertical force at the other end; state the nature of the stresses at any vertical section of the beam.

A uniformly heavy beam is supported at one end; find at what other point in its length it should be supported so that it is least likely to break at any point.

2. A heavy uniform rod, weight  $W$  and length  $2a$ , rests horizontally on two pegs one at each end. A heavy particle whose weight is  $n$  times that of the rod is placed at  $N$ . If  $C$  be the middle point of the rod, show that the bending moment will be greatest either at some point between  $N$  and  $C$  or at  $N$  according as the distance of  $N$  from  $C$  is  $>$  or  $<$   $n$  times its distance from the nearer end of the rod.

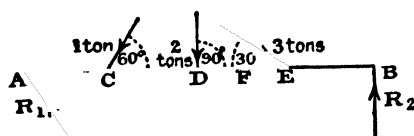


Fig. 167.

3. The diagram shows a beam AB hinged at A and supported at B. Find the reactions  $R_1$  and  $R_2$  due to the given system of loads, and the direction of  $R_1$ . Show the bending moment diagram

and state the value of the bending moment at F.  $AC = CD = DE = EB = 5 \text{ ft.}$ ,  $DF = FE$ .

4. A uniform rod is supported horizontally by two props at equal distances from the ends. Taking suitable numerical values, plot (i) the bending moment at a prop, (ii.) the bending moment at the centre, in each case against the distance between the props. Find from the graphs how the rod should be supported so that it is least likely to break.

5. Explain the terms bending moment, shearing stress, at any point of a rod.

A heavy uniform rod rests horizontally on two pegs, one of which is at one end. Find the point under which the other must be placed that the bending moment at the middle point may be zero.

6. Draw diagrams to represent the shearing stress and bending moment for a horizontal beam, supported at its middle point, with one end fixed and loaded with 30 tons at the other, the weight of the beam being neglected.

7. A simply supported beam has a span of 24 ft. A uniformly distributed load of 1.5 tons per foot of length, 8 ft. long, travels along the beam. Find the maximum bending moment at a section 10 ft. from one of the supports. Prove the method you employ.

8. A beam ABCD, supported at A and D, carries a uniform load from B to C; sketch the diagrams of shearing force and bending moment.

Explain the relations between the discontinuities (if any) in the two diagrams.

9. A heavy uniform beam 16 ft. long is supported at points distant one foot from each end. Draw curves showing the bending moment and shearing stress at all points of the beam.

10. A bar whose section is a square, one cm. in the side rests on two knife-edges a metre apart. A load of 1 Kg. at the centre produces a deflection there of 1.15 cms. Find the value of the modulus of elasticity for the material of the bar.

11. A shaft whose section has a radius  $a$  is transmitting energy at the rate  $P$ , revolving steadily with angular velocity  $\omega$ . Prove that the greatest shearing stress developed in the shaft is

$$\frac{2P}{\pi\omega a^3}.$$

Also show that the least diameter of the shaft in order that the stress may not exceed 4 tons per sq. in., when the shaft is working at 1600 horse-power, and making 25 revolutions per minute is 13.2 in.

12. A disc, radius 2 ins. and weighing 1 lb. is suspended symmetrically by a wire, radius .04 in., and length 1 ft. If the periodic time of rotational oscillation be 20 sec., calculate the coefficient of rigidity of the metal of which the wire is composed.

13. Referring to the previous question, if the coefficient of rigidity be 10,000,000 lbs. per sq. in., calculate the length of the wire for the periodic time to be 1 sec.



14. When the length of the wire of a torsional pendulum is 2 ft. long and a sphere is suspended the periodic time is 10 secs. If now the length be reduced to 1.5 ft. and the sphere be replaced by a cube, the wire being attached to the middle point of one of its faces, find the periodic time given that the length of the edges of the cube equals the diameter of the sphere and that the mass of the cube is 1.156 times that of the sphere.

15. A beam of length  $l$  and weight  $W$  is fixed horizontally at one end and supported at the other so that the two ends are in the same horizontal line. Show that, at the supported end, the beam makes an angle  $\tan^{-1} \frac{Wl^2}{48EI}$  with the horizontal.

16. A uniform beam 10 ft. long is clamped horizontally at both its ends and it is found that the depression at the centre is  $1\frac{1}{8}$  inch. If now a weight equal to twice the weight of the beam be suspended at the middle point, find the depression there.

[N.B. Shear on one half the beam at its middle point = weight of beam.]

17. Referring to the previous question find the depression if the beam were clamped at one end and supported horizontally at the other, no weight being suspended.

18. A uniform beam is 6 ft. long and its cross section is a square whose sides are 2 in. long. It is clamped horizontally at one end and the other end is found to be depressed 4 in. If a pole of the same material, 4 ft. long and cross section a circle of 2 in. diameter, be clamped horizontally at one end, calculate the depression of the other end.

19. A uniform beam 12 ft. long is clamped horizontally at a point 4 ft. from one end. If the depression of this end be 6 in. find the depression of the other end.

20. A uniform girder 14 ft. long has cross section as given in Ex. I., Chapter VII., and its weight is 1 ton. It is supported at its ends in a horizontal position. Given that the modulus of elasticity of the material is  $25 \times 10^6$  lb. wt. sq. in., calculate the depression at the centre.

21. An iron bar, 2 inches in diameter, is bent into the form of an arc of a circle of 400 feet diameter. If the modulus of elasticity is  $29 \times 10^6$  lb. per sq. in., show that the maximum stress at any point of a transverse section is 12080 lbs. per sq. in.

22. If  $\rho$  is the density of the material of a beam of rectangular section with length  $l$ , breadth  $b$ , depth  $d$  simply supported at its ends prove that the maximum bending moment due to its own weight is  $\frac{1}{8} \rho b d l^2$ . If  $\rho = \frac{3}{8}$  lb. per cub. in.,  $d = 15$  in., find the maximum span for a maximum safe stress of 2500 lb. per sq. in.

## CHAPTER XVI

### AN INTRODUCTION TO VECTOR ANALYSIS

**122. Vectors.** A *scalar* is a quantity which has magnitude, but has no direction in space. Thus density, mass, temperature and energy are examples of scalar quantities.

A *vector*, or vector quantity, has both magnitude and direction. Displacement, velocity, acceleration, momentum and force are examples of vectors. A line, having both magnitude and direction, can represent a vector quantity. Any vector can therefore be represented by a straight line with an arrow-head to indicate the direction of the vector. If O and P are the ends of the line and the direction of the vector is from O to P, then the vector can be denoted by  $\overrightarrow{OP}$  or by  $\underline{OP}$ , where OP (without a line) denotes the magnitude of the vector. In text-books, vectors are usually printed in Clarendon type, e.g.  $\mathbf{r}$ , but as this is impossible in written work, it is best to draw a line underneath the quantity and write the vector as  $\underline{r}$ . Then the ordinary letter  $r$ , or italics in printed work, will denote the magnitude of  $\mathbf{r}$ .

Two vectors are said to be *equal* if they have the same direction and are of equal magnitude. This equality will be written  $\mathbf{a} = \mathbf{b}$ .

A force is a vector quantity, but as the effect of a force will depend on its point of application, a force is said to be a *localised vector*. A single vector cannot completely represent the effect of a force acting on a rigid body, since the effect of the force depends on its position.

A zero vector is one whose magnitude is zero. It is denoted by O. A unit vector is a vector of unit magnitude.

Two vectors are added in the same way as two displacements, and vectors therefore obey the parallelogram law. Thus, in Fig. 168  $\mathbf{OC}$  is the sum of  $\mathbf{a}$  and  $\mathbf{b}$ .

If  $\mathbf{OC} = \mathbf{c}$ , then

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

Moreover,  $\mathbf{c} = \mathbf{b} + \mathbf{a}.$

$$\therefore \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a},$$

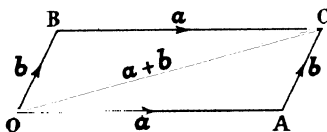


Fig. 168.

so that vector quantities obey the commutative law of algebra. It is readily seen that they also obey the associative law

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c}.$$

The vector denoted by  $-\mathbf{a}$  has the same magnitude as  $\mathbf{a}$ , but is in the opposite direction.

The subtraction of  $\mathbf{b}$  from  $\mathbf{a}$  can be understood as the sum of  $-\mathbf{b}$  and  $\mathbf{a}$ ; i.e.

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Thus in Fig. 168

$$\mathbf{AB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

and

$$\mathbf{BA} = \mathbf{a} - \mathbf{b}.$$

Finally, it can easily be shown geometrically that

$$n(\mathbf{a} + \mathbf{b}) = n\mathbf{a} + n\mathbf{b},$$

where  $n\mathbf{a}$  means a vector in the same direction as  $\mathbf{a}$  and of  $n$  times its magnitude.

**123. Components of a Vector.** A vector is completely determined by giving its components on three co-ordinate axes. These axes may or may not be mutually perpendicular. If mutually perpendicular, the components may be called *resolutes* or *resolved parts*.

Let  $OX, OY, OZ$  be mutually perpendicular axes (Fig. 169), and let  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  be *unit* vectors, coinciding with the  $x, y$  and  $z$  axes respectively. Let  $x, y$  and  $z$  be the projections of a vector  $\mathbf{r}$  on these axes.

Then, by the definition of addition,

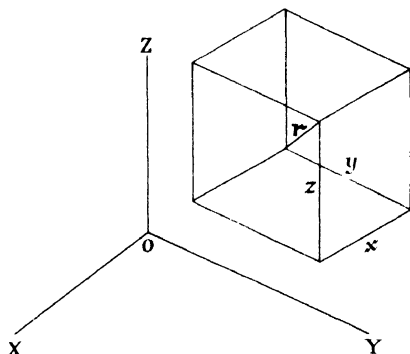


Fig. 169.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad \dots \quad (1)$$

Equation (1) may also be written

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}, \quad \dots \quad (2)$$

where  $r_x = r \cos \alpha$ ,  $\alpha$  being the inclination of  $\mathbf{r}$  to  $Ox$ , etc.

NOTE 1. If  $\mathbf{s} = s_x\mathbf{i} + s_y\mathbf{j} + s_z\mathbf{k}$  and  $\mathbf{r} = \mathbf{s}$ , then  
 $s_x = r_x, s_y = r_y$  and  $s_z = r_z$ .

As an example, equal forces must have equal components in any three mutually perpendicular directions.

NOTE 2. Vectors may be compounded by adding their like components. Thus  
 $\mathbf{r} + \mathbf{s} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}$ ,  
 where  $\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$  and  $\mathbf{s} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ .

### 124. Vector Equations of a Straight Line and of a Plane.

(a) Suppose the line passes through a given point A parallel to a given unit vector  $\mathbf{b}$  whose direction cosines are  $\lambda, \mu, \nu$ . Let  $\mathbf{a}$  be the position vector of A; and let the position vector of P, any point on the line through A, be  $\mathbf{r}$  (Fig. 170). Now  $\mathbf{AP}$  is parallel to the vector  $\mathbf{b}$

$$\therefore \mathbf{AP} = t\mathbf{b},$$

where  $t$  is some number.

Then the equation of the line is

$$\mathbf{r} = \mathbf{a} + t\mathbf{b}. \quad \dots \dots (3)$$

NOTE. If O is taken as the origin of co-ordinate axes, A being  $(x_0, y_0, z_0)$ , P  $(x, y, z)$ , then since (3) is equivalent to the three scalar equations,

$$x = x_0 + t\lambda, \quad y = y_0 + t\mu, \quad z = z_0 + t\nu;$$

$$\text{it follows that} \quad \frac{x - x_0}{\lambda} = \frac{y - y_0}{\mu} = \frac{z - z_0}{\nu} = t. \quad \dots \dots (4)$$

(b) It is given that the line passes through the points A and B, whose position vectors are  $\mathbf{a}$  and  $\mathbf{b}$ .

As before, let P be any point on the line (Fig. 171).

$$\text{Now } \mathbf{AB} = \mathbf{b} - \mathbf{a},$$

$\therefore$  AP is parallel to the vector  $\mathbf{b} - \mathbf{a}$ .

Hence, as in (3) above,

$$\mathbf{r} = \mathbf{a} + k(\mathbf{b} - \mathbf{a}), \quad \dots (5)$$

where  $k$  is some number. Equation

(5) may be written in the form

$$\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b}, \quad \text{where } \lambda + \mu = 1.$$

(c) A plane passes through the point A, whose position vector is  $\mathbf{r}_0$ , the plane being parallel to the two vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

Let P be any point in the plane, the position vector of P being  $\mathbf{r}$ , then the equation of the plane is

$$\mathbf{r} = \mathbf{r}_0 + k\mathbf{a} + l\mathbf{b}. \quad \dots \dots (6)$$

The corresponding scalar equations will be

$$x = x_0 + ka_x + lb_x, \quad y = y_0 + ka_y + lb_y, \quad z = z_0 + ka_z + lb_z.$$

On elimination of  $k$  and  $l$  from these three equations we obtain

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0 \quad \dots (7)$$

as the Cartesian equation of the plane.

**125. To find the Point which divides the Line joining two Points in a given Ratio.** Let  $\mathbf{a}, \mathbf{b}$  be position vectors drawn

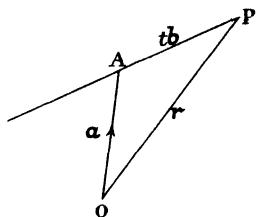


Fig. 170.

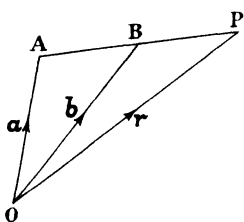


Fig. 171.

from O to A and B respectively, and let the point R divide the line AB in the ratio  $m : n$  (Fig. 172).

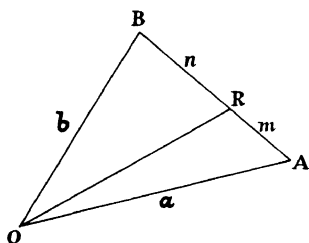


Fig. 172.

$$\text{Now } \vec{AB} = \vec{b} - \vec{a},$$

$$\therefore \vec{AR} = \frac{m}{m+n} (\vec{b} - \vec{a})$$

then

$$\begin{aligned} \vec{OR} &= \vec{a} + \frac{m}{m+n} (\vec{b} - \vec{a}) \\ &= \frac{n\vec{a} + m\vec{b}}{m+n} \quad \dots (8) \end{aligned}$$

$$\text{or } (m+n)\vec{OR} = n\vec{a} + m\vec{b}.$$

NOTE. From (8) we can deduce the usual values of the Cartesian co-ordinates of R.

If A is  $(x_1, y_1, z_1)$ , B  $(x_2, y_2, z_2)$  and R  $(\bar{x}, \bar{y}, \bar{z})$  with respect to co-ordinates axes through O,

$$\bar{x} = \frac{nx_1 + mx_2}{m+n}, \quad \bar{y} = \frac{ny_1 + my_2}{m+n}, \text{ etc.} \quad \dots \dots (9)$$

**Ex.** If two concurrent forces are represented by  $n\vec{OA}$  and  $m\vec{OB}$  respectively, their resultant is given by  $(m+n)\vec{OR}$ , where R divides AB so that  $AR : RB = m : n$ .

From Fig. 172 above

$$\vec{OA} = \vec{OR} + \vec{RA},$$

$$\vec{OB} = \vec{OR} + \vec{RB}.$$

$\therefore$  the resultant of  $n\vec{OA}$  and  $m\vec{OB}$

$$\begin{aligned} &= (m+n)\vec{OR} + n\vec{RA} + m\vec{RB} \\ &= (m+n)\vec{OR}, \quad \dots \dots \dots (10) \end{aligned}$$

since

$$n\vec{RA} = -n\vec{AR} = -m\vec{RB}.$$

*Application to centre of mass.* If G is the centre of mass of particles  $m_1, m_2$  at  $P_1, P_2$ , then G divides  $P_1P_2$  in the ratio  $m_2 : m_1$ .

Hence if  $\vec{p}_1, \vec{p}_2, \vec{OG}$  are the position vectors of  $m_1, m_2$  and G, we obtain from (8)

$$\vec{OG} = \frac{m_1\vec{p}_1 + m_2\vec{p}_2}{m_1 + m_2}$$

or

$$(m_1 + m_2)\vec{OG} = m_1\vec{p}_1 + m_2\vec{p}_2.$$

Similarly, if G is the centre of mass of particles  $m_1, m_2, m_3, \dots$  at  $P_1, P_2, P_3, \dots$  and  $M = \sum m$

$$M.\vec{OG} = \sum m\vec{p}. \quad \dots \dots (11)$$

If  $\vec{OG} = \bar{x}\vec{i} + \bar{y}\vec{j} + \bar{z}\vec{k}$ ,  $\vec{p} = \vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}$ ,  
 then  $M\bar{x} = \sum mx$ ,  $M\bar{y} = \sum my$ ,  $M\bar{z} = \sum mz$ . (11a)

*The centroid is independent of the origin O*

Let  $O'$  be a point whose position vector relative to  $O$  is  $\mathbf{a}$ , then for this origin (Fig. 173)

$$\mathbf{p}'_1 = \mathbf{p}_1 - \mathbf{a} ; \quad \mathbf{p}'_2 = \mathbf{p}_2 - \mathbf{a}, \text{ etc.}$$

So that the centroid  $G'$  is given by

$$\begin{aligned} \mathbf{O'G'} &= \frac{\sum m \mathbf{p}'}{\sum m} = \frac{\sum m (\mathbf{p}_1 - \mathbf{a})}{\sum m} \\ &= \frac{\sum m_1 \mathbf{p}_1}{\sum m} - \mathbf{a} \\ &= \mathbf{OG} - \mathbf{a} = \mathbf{O'G}. \end{aligned}$$

$\therefore G$  and  $G'$  coincide.

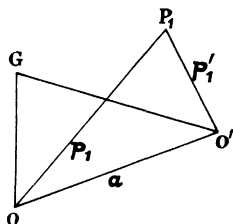


Fig. 173.

*Application to concurrent forces*

Let a number of concurrent forces act at a point  $O$ , through  $P_1, P_2, P_3$ , etc., where the force in  $OP_1$  is  $m_1 \mathbf{OP}_1$ , etc. Then, if  $\mathbf{R}$  is the resultant of the forces,

$$\begin{aligned} \mathbf{R} &= \sum m \cdot \mathbf{OP} = \sum m \mathbf{p} \\ &= \mathbf{OG} \cdot \sum m, \quad \dots \quad \dots \quad \dots \quad (12) \end{aligned}$$

where  $G$  is the centroid of masses  $m_1, m_2$ , etc., at  $P_1, P_2$ , etc., respectively and  $\sum m = M$ .

If the forces are in equilibrium  $G$  will coincide with  $O$ .

**126. Derivative of a Vector.** *Definition.* If a vector  $\mathbf{r}$  is a function of a scalar variable  $t$ , and  $\mathbf{r}$  becomes  $\mathbf{r} + \delta \mathbf{r}$  when  $t$  becomes  $t + \delta t$ , then

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta \mathbf{r}}{\delta t}.$$

Note that  $\frac{d\mathbf{r}}{dt}$  is a vector.

Suppose that a particle is at a point  $P$ , whose position vector is  $\mathbf{r}$  at time  $t$ , and that at time  $t + \delta t$  the particle is at  $P'$ , whose position vector is  $\mathbf{r} + \delta \mathbf{r}$  (Fig. 174).

Then  $PP' = \delta \mathbf{r}$

and  $\frac{d\mathbf{r}}{dt}$  = velocity of  $P$  relative to  $O$  and is in the direction of the tangent to the path of  $P$ . This can be written  $\dot{\mathbf{r}}$ .

Similarly,  $\frac{d^2\mathbf{r}}{dt^2}$  = acceleration of  $P$  relative to  $O$ , and may be written  $\ddot{\mathbf{r}}$ , a vector representing the acceleration both in magnitude and direction.

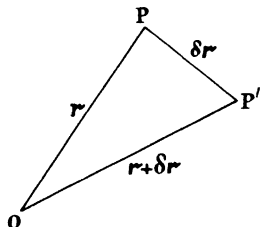


Fig. 174.

**127. Rules for Differentiation.**

(1) If  $\mathbf{c} = \mathbf{a} + \mathbf{b}$ , then  $\frac{d\mathbf{c}}{dt} = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$ . .. (13)

(2) If  $\mathbf{c} = x\mathbf{a}$ , where  $x$  is a scalar,  
 then  $\delta\mathbf{c} = \delta(x\mathbf{a}) = (x + \delta x)(\mathbf{a} + \delta\mathbf{a}) - x\mathbf{a}$   
 $= \mathbf{a}\delta x + x\delta\mathbf{a} + \delta x\delta\mathbf{a}$ .  
 $\therefore \frac{d\mathbf{c}}{dt} = \mathbf{a} \frac{dx}{dt} + x \frac{d\mathbf{a}}{dt}$ . .. (14)

(3) If  $\mathbf{r}$  is a continuous function of a scalar variable  $s$ , and  $s$  is a continuous function of  $t$ , then identically

$$\frac{\delta\mathbf{r}}{\delta t} = \frac{\delta\mathbf{r}}{\delta s} \cdot \frac{\delta s}{\delta t}$$

$$\therefore \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \cdot \frac{ds}{dt} \quad (15)$$

**128. Differentiation of a Unit Vector.** Let  $\mathbf{OR} = \mathbf{R}$  be the position of a *unit* vector at time  $t$  (Fig. 175).  $Ox$  is a fixed direction and  $\angle ROx = \theta$ . Let  $\mathbf{OR}$  become  $\mathbf{OR}'$  at time  $t + \delta t$ . Then the limiting direction of  $\mathbf{RR}'$  is tangential to the circle at  $R$ .

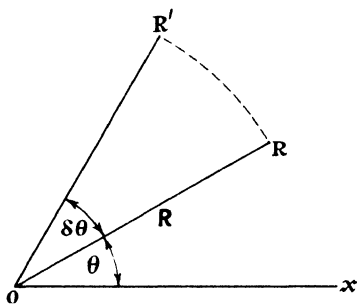


Fig. 175.

Now  $\frac{d\mathbf{R}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{RR}'}{\delta t}$ .

But the numerical value of this limit is  $\frac{d\theta}{dt}$  since  $OR = \text{unity}$ ,

hence  $\dot{\mathbf{R}} = \dot{\theta}\mathbf{s}$ , .. .. (16)

where  $\mathbf{s}$  is a unit vector perpendicular to  $\mathbf{R}$ , in the plane of motion and in the direction of  $\theta$  increasing.

**NOTE.** If  $\mathbf{R}$  is fixed in direction,  $\dot{\mathbf{R}} = \mathbf{0}$ .

Since  $\mathbf{s}$  is perpendicular to  $\mathbf{R}$ , the angle it makes with  $Ox$  is  $\frac{\pi}{2} + \theta$ ; also  $\dot{\mathbf{s}}$  is perpendicular to  $\mathbf{s}$ , i.e. parallel to  $-\mathbf{R}$ ,

$$\therefore \dot{\mathbf{s}} = -\dot{\theta}\mathbf{R}. \quad \dots \dots (17)$$

**129. Radial and Transversal Components of Velocity and Acceleration.** Let the path of a moving particle  $P$  be a *plane* curve. Let  $OP$  be a vector of magnitude  $r$  in the direction of the unit vector  $\mathbf{R}$ ; then

$$\begin{aligned} \mathbf{OP} &= r\mathbf{R}. \\ (a) \text{ Velocity of } P &= \dot{\mathbf{OP}} = \dot{r}\mathbf{R} + r\dot{\mathbf{R}} \quad \dots \text{ from (14)} \\ &= \dot{r}\mathbf{R} + r(\dot{\theta}\mathbf{s}). \quad \dots \text{ from (16)} \end{aligned}$$

Now  $\mathbf{R}$  and  $\mathbf{s}$  are perpendicular unit vectors, hence the radial and transversal components of the velocity of P are  
 $\dot{r}$  and  $r\dot{\theta}$ .

$$(b) \text{ Acceleration of P} = \ddot{\mathbf{O}}\mathbf{P} = \frac{d}{dt}(\dot{\mathbf{O}}\mathbf{P}) = \frac{d}{dt}(\dot{r}\mathbf{R} + r\dot{\theta}\mathbf{s})$$

$$= \ddot{r}\mathbf{R} + \dot{r}\dot{\mathbf{R}} + \dot{r}\dot{\theta}\mathbf{s} + \mathbf{s} \frac{d}{dt}(r\dot{\theta}) - r\dot{\theta}^2\mathbf{R}, \quad \text{from (17)}$$

$$= (\ddot{r} - r\dot{\theta}^2)\mathbf{R} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{s}. \quad \dots \dots (18)$$

Hence the *radial* acceleration  $= \ddot{r} - r\dot{\theta}^2$   
 and the *transverse* acceleration  $= 2\dot{r}\dot{\theta} + r\ddot{\theta}$   
 $= \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}).$

**130. Tangential and Normal Components of Velocity and Acceleration.** Let  $(s, \psi)$  be the intrinsic co-ordinates of P. Let  $\mathbf{S}$  and  $\mathbf{N}$  be unit vectors in the directions of the tangent and inward drawn normal respectively.

It has been proved that the instantaneous velocity of P is tangential to the path of P. Its magnitude is, by definition,  $\frac{ds}{dt}$ ,

$$\therefore \text{velocity of P} = \dot{s}\mathbf{S} \quad \dots \dots (19)$$

and there is no normal component.

$$\text{Also, acceleration of P} = \frac{d}{dt}(\dot{s}\mathbf{S}) = \ddot{s}\mathbf{S} + \dot{s}\dot{\mathbf{S}}$$

$$= \ddot{s}\mathbf{S} + \dot{s}\dot{\psi}\mathbf{N}, \text{ since } \dot{\mathbf{S}} = \dot{\psi}\mathbf{N}. \dots (20)$$

Hence (i) the tangential component of the acceleration  $= \ddot{s}$ ,

(ii) the normal component  $= \dot{s}\dot{\psi} = \dot{s}^2/\rho$ .

**131. Scalar Products.** The *scalar product* of two vectors  $\mathbf{a}$  and  $\mathbf{b}$  is written

$$\mathbf{a} \cdot \mathbf{b}$$

(and therefore often called the *dot product*)

It is defined by the equation

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta = \mathbf{b} \cdot \mathbf{a}, \quad \dots \dots (21)$$

where  $\theta$  is the angle between the directions of the two vectors. Note that  $b \cos \theta$  is the measure of the length of the projection of  $\mathbf{b}$  in the direction of  $\mathbf{a}$ . Hence  $\mathbf{a} \cdot \mathbf{b}$  is the product of two numbers and is a scalar number.

If two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular, since  $\cos \theta = 0$ ,

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

$$\text{Thus } \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0. \quad \dots \dots (22)$$

If  $\mathbf{b} = \mathbf{a}$ ,  $\therefore \cos \theta = 1$ .

$$\therefore \mathbf{a} \cdot \mathbf{a} = a^2 \text{ (a being the magnitude of } \mathbf{a} \text{).}$$

$$\text{Thus } \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1. \quad \dots \dots (23)$$



If  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are three vectors, then  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

for  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \text{projection of } (\mathbf{b} + \mathbf{c}) \text{ on } \mathbf{a}$

$$= \mathbf{a} \times \text{projection of } \mathbf{b} + \mathbf{a} \times \text{projection of } \mathbf{c}$$

$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \quad \dots \quad (24)$$

Hence also  $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} + \mathbf{b})$ .

Suppose further that  $\mathbf{a}, \mathbf{b}$  are expressed in terms of the unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ . Then, with the usual notation,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \\ &= a_x b_x + a_y b_y + a_z b_z \quad \dots \quad (25) \end{aligned}$$

from (22) and (23).

$$\text{Hence } \cos \theta = (a_x b_x + a_y b_y + a_z b_z) / ab,$$

and, as the direction cosines  $l, m, n$  of  $\mathbf{a}$  are  $a_x/a, a_y/a, a_z/a$ , and the direction cosines  $l', m', n'$  of  $\mathbf{b}$  are  $b_x/b, b_y/b, b_z/b$ ,

$$\therefore \cos \theta = ll' + mm' + nn'.$$

NOTE. Using (25), putting  $\mathbf{b} = \mathbf{a}$ , and defining  $\mathbf{a}^2$  as  $\mathbf{a} \cdot \mathbf{a}$ ,  
 $\mathbf{a}^2 = a^2 = (a_x)^2 + (a_y)^2 + (a_z)^2$ ,  
 i.e.  $OP^2 = x^2 + y^2 + z^2$ , where P is  $(x, y, z)$ .

### 132. Use of Scalar Products.

1. To prove that the altitudes of a triangle are concurrent.

Let the triangle be ABC, and let the altitudes from A and B meet in O.

Let  $\mathbf{OA} = \mathbf{a}$ ,  $\mathbf{OB} = \mathbf{b}$ ,

$\mathbf{OC} = \mathbf{c}$ , then  $\mathbf{BC} = \mathbf{c} - \mathbf{b}$ ,

$\mathbf{CA} = \mathbf{a} - \mathbf{c}$ ,  $\mathbf{BA} = \mathbf{a} - \mathbf{b}$ .

Since OA is perpendicular to BC,  $\mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0$ ,

$$\text{i.e. } \mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}.$$

Since OB is perpendicular to CA,  $\mathbf{b} \cdot (\mathbf{a} - \mathbf{c}) = 0$ .

$$\therefore \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c}.$$

Hence,  $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$  or  $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0$ , and it therefore follows that OC is perpendicular to AB, which proves the proposition.

NOTE. If OABC is a tetrahedron, the above argument proves that if two pairs of opposite edges of the tetrahedron are perpendicular so also is the third pair.

2. To prove that the angle between two lines with direction cosines  $(l_1, m_1, n_1)$  and  $(l_2, m_2, n_2)$  is  $\cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)$ .

Consider unit vectors  $\mathbf{u}_1, \mathbf{u}_2$  at angle  $\theta$  to one another, where

$$\mathbf{u}_1 = l_1 \mathbf{i} + m_1 \mathbf{j} + n_1 \mathbf{k}$$

and

$$\mathbf{u}_2 = l_2 \mathbf{i} + m_2 \mathbf{j} + n_2 \mathbf{k}.$$

Then  $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 1 \cdot \cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ , from (25)

i.e.  $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ .

If the lines are perpendicular,  $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ .

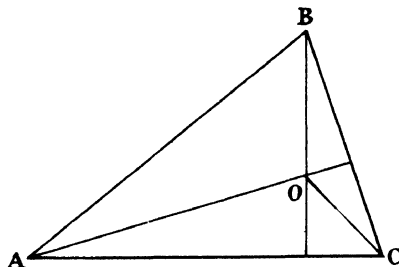


Fig. 176.

### 3. To find the resultant of two concurrent forces.

Let  $\mathbf{OP}$  and  $\mathbf{OQ}$  be the forces at angle  $\alpha$ , and  $\mathbf{OR}$  their resultant (Fig. 177).

Then (i)  $\mathbf{OR} = \mathbf{OP} + \mathbf{OQ}$ ;

$$(ii) \quad \mathbf{OR} \cdot \mathbf{OR} = (\mathbf{OP} + \mathbf{OQ}) \cdot (\mathbf{OP} + \mathbf{OQ}) \\ = \mathbf{OP} \cdot \mathbf{OP} + \mathbf{OQ} \cdot \mathbf{OQ} + 2 \cdot \mathbf{OP} \cdot \mathbf{OQ}.$$

$$\therefore OR^2 = OP^2 + OQ^2 + 2OP \cdot OQ \cos \alpha.$$

NOTE. Let  $\mathbf{OP}_1, \mathbf{OP}_2, \mathbf{OP}_3, \dots, \mathbf{OP}_n$  be any number of concurrent forces, and  $\mathbf{OR}$  their resultant; then

$$\mathbf{OR} = \mathbf{OP}_1 + \mathbf{OP}_2 + \dots + \mathbf{OP}_n$$

$$\therefore OR^2 = \left( \sum_{r=1}^n \mathbf{OP}_r \right)^2 \\ = \sum_{r=1}^n \mathbf{OP}_r^2 + 2 \sum \mathbf{OP}_r \cdot \mathbf{OP}_s \\ = \sum_{r=1}^n OP_r^2 + 2 \sum OP_r OP_s \cos \theta_s$$

( $r = 1, 2, \dots, n$ ;  $t < s$ ;  $s = 2, 3, \dots, n$ ), where  $\theta_s$  is the angle between  $\mathbf{OP}_r$  and  $\mathbf{OP}_s$ .

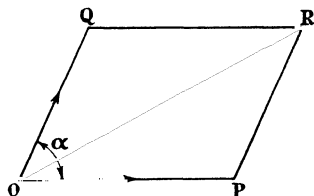


Fig. 177.

### 4. Work done by a force.

The work done is the product of the force, and the distance moved in the direction of the force, so work  $= Fd \cos \theta = \mathbf{F} \cdot \mathbf{d}$ , if  $F$  is constant and  $\theta$  is the angle between  $F$  and the displacement  $d$ .

**Ex.** Three forces  $l\mathbf{OA}$ ,  $m\mathbf{OB}$ ,  $n\mathbf{OC}$  act at a point  $O$ . If  $A$ ,  $B$ ,  $C$  are the vertices of a triangle and  $l, m, n$  are any three numbers such that  $l + m + n = 0$ , prove

(i) that their resultant  $R$  is given by  $R^2 = -mnBC^2 - nlCA^2 - lmAB^2$ ;

(ii) that the direction of the resultant is that of the straight line joining  $A$  to the point dividing  $BC$  in the ratio  $n : m$ .

Part (i) of this problem may be solved directly by using the note in (3) above. Both parts can, however, be dealt with as follows (Fig. 178):

$$\mathbf{R} = l\mathbf{OA} + m\mathbf{OB} + n\mathbf{OC} \\ = l\mathbf{OA} + m(\mathbf{OA} + \mathbf{AB}) + n(\mathbf{OA} + \mathbf{AC}) \\ = (l + m + n)\mathbf{OA} + m\mathbf{AB} + n\mathbf{AC}$$

$$= m\mathbf{AB} + n\mathbf{AC}, \text{ since } l + m + n = 0,$$

and, using §125, this proves that the resultant divides  $BC$  in the ratio  $n : m$ .

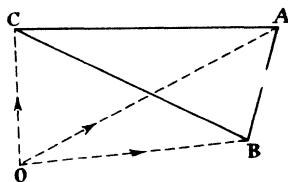


Fig. 178.

$$\begin{aligned}
 \text{Then } \mathbf{R}^2 = R^2 &= m^2 AB^2 + n^2 AC^2 + 2mn AB \cdot AC \cos BAC \\
 &= m^2 AB^2 + n^2 AC^2 + mn(AB^2 + AC^2 - BC^2) \\
 &= m(m+n)AB^2 + n(n+m)AC^2 - mnBC^2 \\
 &= -mlAB^2 - nlAC^2 - mnBC^2.
 \end{aligned}$$

### 133. Derivative of a Scalar Product.

$$\begin{aligned}
 \text{Now } \delta(\mathbf{a} \cdot \mathbf{b}) &= (\mathbf{a} + \delta\mathbf{a}) \cdot (\mathbf{b} + \delta\mathbf{b}) - \mathbf{a} \cdot \mathbf{b} \\
 &= \mathbf{b} \cdot \delta\mathbf{a} + \mathbf{a} \cdot \delta\mathbf{b} \dots \quad (\text{a scalar quantity}),
 \end{aligned}$$

$$\therefore \frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{b} \cdot \frac{d\mathbf{a}}{dt} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt} \dots \quad (\text{a scalar quantity}). \quad (26)$$

$$\text{If } \mathbf{b} = \mathbf{a}, \quad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt}.$$

$$\begin{aligned}
 \text{But } \mathbf{a} \cdot \mathbf{a} &= a^2, \quad \text{and} \quad \frac{d}{dt}(a^2) = 2a \frac{da}{dt}. \\
 \therefore \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) &= 2a \frac{da}{dt} \quad \dots \quad \dots \quad (27)
 \end{aligned}$$

and, if  $\mathbf{a}$  is a vector constant in length,  $\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$ . This shows that  $\mathbf{a}$  and  $\frac{d\mathbf{a}}{dt}$  are perpendicular to one another.

NOTE.  $\frac{d}{dt}(\mathbf{a} \cdot \mathbf{b})$  can easily be obtained by differentiating the scalar result given in (25).

### 134. Work and Energy.

From the definition of work in §132, it follows that for an infinitesimal displacement  $\delta\mathbf{r}$  of a particle, the work done by a force  $\mathbf{F}$  acting on it is  $\mathbf{F} \cdot \delta\mathbf{r}$ . If this displacement takes place in time  $\delta t$ , we see that the instantaneous rate of working is

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v}. \quad \dots \quad \dots \quad (28)$$

Now the kinetic energy (T) of the particle, of mass  $m$ , when moving with velocity  $\mathbf{v}$  is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}^2. \quad \dots \quad \dots \quad (29)$$

The rate of increase of kinetic energy is

$$\frac{dT}{dt} = \frac{d}{dt}(\frac{1}{2}m\mathbf{v}^2) = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}. \quad \dots \quad \text{from (27)}.$$

But  $\frac{d\mathbf{v}}{dt}$  is the acceleration of the particle and  $m \frac{d\mathbf{v}}{dt} = \mathbf{F}$ , where  $\mathbf{F}$  is the resultant of the forces acting on the particle.

$$\therefore \frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}, \quad \dots \quad \dots \quad (30)$$

i.e. the rate of increase of the kinetic energy of the particle is equal to the rate at which  $\mathbf{F}$  is doing work on the particle.

**135. Vector Products.** Let  $\mathbf{a}$ ,  $\mathbf{b}$  be two vectors inclined at an angle  $\theta$  and let  $\mathbf{n}$  be a unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ , measured in such a sense that if  $\mathbf{a}$  is rotated towards  $\mathbf{b}$  through an angle  $\theta$ , then a right-handed screw would move forward (Fig. 179.) The *vector product* of  $\mathbf{a}$  and  $\mathbf{b}$  is then defined as equal to  $ab \sin \theta \mathbf{n}$ , a vector quantity.

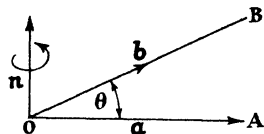


Fig. 179

The vector product may be denoted by

$$\mathbf{a} \times \mathbf{b} \text{ (and so called the cross product)} \\ \text{or } \nabla \mathbf{ab} \text{ or } \mathbf{a} \wedge \mathbf{b} \text{ or } [\mathbf{ab}].$$

$$\text{Hence} \quad \mathbf{a} \wedge \mathbf{b} \equiv ab \sin \theta \mathbf{n}. \quad \dots \dots \dots (31)$$

We may say that  $\mathbf{a} \wedge \mathbf{b}$  is  $a$  multiplied by the component of  $\mathbf{b}$  in the plane perpendicular to  $\mathbf{a}$  turned through a right angle about  $\mathbf{a}$  in the right-handed sense. Note that  $ab \sin \theta = 2$  triangle AOB, and this is the numerical value of the vector product.

From the definition,  $\mathbf{b} \wedge \mathbf{a}$  must be drawn in the opposite direction to  $\mathbf{a} \wedge \mathbf{b}$

$$\therefore \mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}, \quad \dots \dots (32)$$

i.e. the order is not commutative as with a scalar product. If  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  give three mutually perpendicular directions constituting a right-handed frame, then, from the definition

$$\mathbf{i} \wedge \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \wedge \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \wedge \mathbf{i} = \mathbf{j}, \quad \mathbf{i} \wedge \mathbf{k} = -\mathbf{j}, \text{ etc.} \quad (33)$$

If  $\mathbf{a}$  and  $\mathbf{b}$  are parallel,  $\sin \theta = 0$ ,  $\therefore \mathbf{a} \wedge \mathbf{b} = 0$ , from which we see that

$$\mathbf{a} \wedge \mathbf{a} = 0,$$

$$\text{and} \quad \mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{k} \wedge \mathbf{k} = 0. \quad \dots (34)$$

As with the scalar product, it can be shown that

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}.$$

The proof follows readily from the interpretation following (31). Let  $\mathbf{b}'$ ,  $\mathbf{c}'$  be the projections of  $\mathbf{b}$ ,  $\mathbf{c}$  on the plane perpendicular to  $\mathbf{a}$ , then  $\mathbf{a}$  is perpendicular to both  $\mathbf{b}'$  and  $\mathbf{c}'$ . Also  $\mathbf{b}' + \mathbf{c}'$  is the projection of  $\mathbf{b} + \mathbf{c}$  on that plane.

Now  $\mathbf{a} \wedge \mathbf{b}'$  is of length  $ab'$  and is perpendicular to  $\mathbf{b}'$ , and  $\mathbf{a} \wedge \mathbf{c}'$  is of length  $ac'$  and is perpendicular to  $\mathbf{c}'$ . Both lie in the plane perpendicular to  $\mathbf{a}$  and hence also  $\mathbf{a} \wedge \mathbf{b}' + \mathbf{a} \wedge \mathbf{c}'$ . But the length of this last vector is  $a(b' + c')$  and the vector is perpendicular to  $\mathbf{b}' + \mathbf{c}'$ .

Since  $\mathbf{a} \wedge \mathbf{b}' = \mathbf{a} \wedge \mathbf{b}$  and  $\mathbf{a} \wedge \mathbf{c}' = \mathbf{a} \wedge \mathbf{c}$ , we have

$$\mathbf{a} \wedge \mathbf{b}' + \mathbf{a} \wedge \mathbf{c}' = \mathbf{a} \wedge (\mathbf{b}' + \mathbf{c}').$$

$$\therefore \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}). \quad \dots (35)$$

We deduce from this proposition that

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{c} + \mathbf{d}) = \mathbf{a} \wedge \mathbf{c} + \mathbf{a} \wedge \mathbf{d} + \mathbf{b} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{d}. \quad (36)$$

**136. To express a Vector Product  $\mathbf{a} \wedge \mathbf{b}$  in Terms of its  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  Components.**

$$\mathbf{a} \wedge \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \wedge (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}).$$

Using the distributive law above and remembering (33) and (34)

$$\begin{aligned} \mathbf{a} \wedge \mathbf{b} &= (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k} \\ &= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} \quad \dots \quad \dots \quad (37) \end{aligned}$$

**137. Moment of a Force.** Let  $\mathbf{F}$  be a localised vector acting at  $P(\mathbf{r})$ . The magnitude of its moment about a point  $O$  is  $F$  multiplied by  $ON$ , where  $ON$  is the perpendicular from  $O$  on the line of action of  $\mathbf{F}$  (Fig. 180).

From the definition of a vector product this moment is the numerical value of  $\mathbf{r} \wedge \mathbf{F}$ , where  $OP = \mathbf{r}$ .

This may be written

$$|\mathbf{r} \wedge \mathbf{F}| = \text{moment of } \mathbf{F}.$$

The vector product can, as usual, be represented by a vector perpendicular to the plane of  $\mathbf{r}$  and  $\mathbf{F}$ .

If several forces  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \dots$  are acting at  $P$ , their resultant will be  $\mathbf{R} = \Sigma \mathbf{F}$ , and the moment of this resultant about  $O$  is

$$\begin{aligned} \mathbf{r} \wedge \mathbf{F} &= \mathbf{r} \wedge (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \dots) \\ &= \mathbf{r} \wedge \mathbf{F}_1 + \mathbf{r} \wedge \mathbf{F}_2 + \dots \quad \dots \quad (38) \end{aligned}$$

Hence the vector moment of the resultant is equal to the vector sum of the moments of the component forces.

**138. Angular Velocity as a Vector.** Let  $P$  be a point fixed in a rigid body which is rotating about an axis  $ON$ , with angular velocity  $\omega$  (Fig. 181). This angular velocity can be specified by a vector  $\boldsymbol{\omega}$  of magnitude  $\omega$ , parallel to  $ON$ , whose positive direction is that relative to the rotation.

Let  $OP = \mathbf{r}$  and let  $\mathbf{v}$  be the linear velocity of  $P$ .

Now the magnitude of  $\mathbf{v}$  is  $\omega PT$ , where  $PT$  is the perpen-

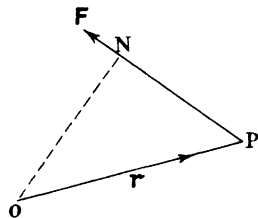


Fig. 180.

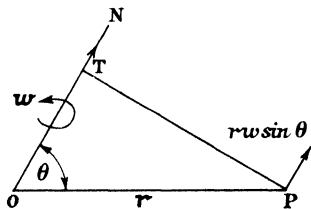


Fig. 181.

dicular on the axis ON. Its direction is perpendicular to the plane NOP. But  $PT = OP \sin \theta$ , where  $\theta$  is the angle between OP and ON. It therefore satisfies the condition for the vector product  $\mathbf{w} \wedge \mathbf{r}$

$$\therefore \mathbf{v} = \mathbf{w} \wedge \mathbf{r}. \quad \dots \quad (39)$$

Note that the direction of this vector is parallel to the velocity of P.

### 139. Derivative of a Vector Product.

Let  $\mathbf{r} \wedge \mathbf{s}$  be the vector product.

$$\begin{aligned} \delta(\mathbf{r} \wedge \mathbf{s}) &= (\mathbf{r} + \delta\mathbf{r}) \wedge (\mathbf{s} + \delta\mathbf{s}) - \mathbf{r} \wedge \mathbf{s} \\ &= \mathbf{r} \wedge \delta\mathbf{s} + \delta\mathbf{r} \wedge \mathbf{s} + \delta\mathbf{r} \wedge \delta\mathbf{s}. \end{aligned}$$

$$\therefore \frac{d}{dt}(\mathbf{r} \wedge \mathbf{s}) = \mathbf{r} \wedge \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \wedge \mathbf{s}. \quad \dots \quad (40)$$

It is simpler, however, when differentiating the cross product, to find  $\mathbf{r} \wedge \mathbf{s}$  first, as in (37).

$$\text{Then} \quad \frac{d}{dt}(\mathbf{r} \wedge \mathbf{s}) = \Sigma \frac{d}{dt}(r_y s_z - r_z s_y) \mathbf{i}. \quad \dots \quad (41)$$

It can then easily be verified by using

$$\mathbf{r} = r_x \mathbf{i} + r_y \mathbf{j} + r_z \mathbf{k}, \quad \mathbf{s} = s_x \mathbf{i} + s_y \mathbf{j} + s_z \mathbf{k},$$

that the right-hand sides of (40) and (41) are equal.

$$\text{If} \quad \mathbf{s} = \frac{d\mathbf{r}}{dt},$$

$$\text{then} \quad \frac{d}{dt} \left( \mathbf{r} \wedge \frac{d\mathbf{r}}{dt} \right) = \mathbf{r} \wedge \frac{d^2\mathbf{r}}{dt^2}, \quad \dots \quad (42)$$

$$\text{since} \quad \frac{d\mathbf{r}}{dt} \wedge \frac{d\mathbf{r}}{dt} = 0.$$

NOTE. Both the scalar and vector products are differentiated as in algebraic products, paying no attention to the dot or  $\wedge$  sign. Care, however, must be taken to preserve the order in the vector product.

**140. Angular Momentum.** Let a moving particle of mass  $m$ , be at P, where  $OP = \mathbf{r}$ , and let its velocity be  $\mathbf{v}$ . The momentum of the particle is  $m\mathbf{v}$ , and the moment of this momentum, or angular momentum about O, is  $\mathbf{r} \wedge m\mathbf{v}$ . Denoting angular momentum by  $\mathbf{H}$ , we have

$$\mathbf{H} = \mathbf{r} \wedge m\mathbf{v}. \quad \dots \quad (43)$$

$$\begin{aligned} \text{Also} \quad \frac{d\mathbf{H}}{dt} &= \dot{\mathbf{r}} \wedge m\mathbf{v} + \mathbf{r} \wedge m \frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \wedge m\mathbf{v} + \mathbf{r} \wedge m\dot{\mathbf{v}}, \end{aligned}$$

where  $\dot{\mathbf{v}}$  is the acceleration of the particle. But  $\mathbf{v} \wedge m\mathbf{v} = 0$ ,

$$\therefore \quad \frac{d\mathbf{H}}{dt} = \mathbf{r} \wedge \mathbf{F}, \quad \dots \quad (44)$$

where  $\mathbf{F}$  is the resultant force acting on the particle. Hence,

from (38), the *rate of increase of the angular momentum of the particle about O is equal to the sum of the moments of the forces acting on the particle about O.*

NOTE. If, as in a central orbit, the particle is acted on by a force towards O,  $\frac{d\mathbf{H}}{dt} = \mathbf{0}$  and therefore  $\mathbf{H}$  is constant in magnitude and direction; i.e. the particle must move in a *plane curve*.

#### 141. Products of three Vectors.

Since  $\mathbf{b} \wedge \mathbf{c}$  is a vector, both a scalar and a vector product can be formed with it and another vector  $\mathbf{a}$ .

The product  $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$  is called the *triple scalar product*, and  $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$  the *triple vector product*, the former being a number and the latter a vector.

The *triple scalar product* represents the volume of a parallelepiped with edges  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ , since  $\mathbf{b} \wedge \mathbf{c} = bc \sin \theta \mathbf{n}$ , where  $\mathbf{n}$  is a unit vector perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$ .

If  $\phi$  is the angle between  $\mathbf{n}$  and  $\mathbf{a}$ , then

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = abc \sin \theta \cos \phi.$$

If A is  $(x_1, y_1, z_1)$ , B  $(x_2, y_2, z_2)$ , C  $(x_3, y_3, z_3)$  then

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad \dots \quad (45)$$

Since the volume remains the same, no matter which face is considered as base,

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \wedge \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \wedge \mathbf{a}) = (\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c}, \text{ etc.} \quad (46)$$

Thus the signs of multiplication may be interchanged, so long as the cyclic order remains the same.

If  $\mathbf{OA}$ ,  $\mathbf{OB}$ ,  $\mathbf{OC}$  are three vectors, then

$$\mathbf{OA} \cdot (\mathbf{OB} \wedge \mathbf{OC}) = \pm 6 \text{ tetrahedron OABC}, \quad (47)$$

for  $\mathbf{OB} \wedge \mathbf{OC} = 2\Delta\text{OBC}\mathbf{n}$

(see §135) (Fig. 182) and

$$\begin{aligned} \mathbf{OA} \cdot (\mathbf{OB} \wedge \mathbf{OC}) \\ = 2\Delta\text{OBC} \cdot \text{OA} \cos \phi, \end{aligned}$$

where  $\phi$  is the angle between  $\mathbf{OA}$  and  $\mathbf{n}$ ,

$$\therefore \mathbf{OA} \cdot (\mathbf{OB} \wedge \mathbf{OC})$$

$$= 2h\Delta\text{OBC},$$

where  $h$  is the altitude,  $= \pm 6 \text{ tetrahedron OABC}.$

The product is positive if

$\mathbf{OA}$ ,  $\mathbf{OB}$ ,  $\mathbf{OC}$  is a right-handed triad and negative if otherwise.

NOTE 1. If  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  are coplanar,  $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = 0$ .

NOTE 2. If any two of the three vectors are equal  $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = 0$ .

NOTE 3. The above product may be denoted by  $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ .

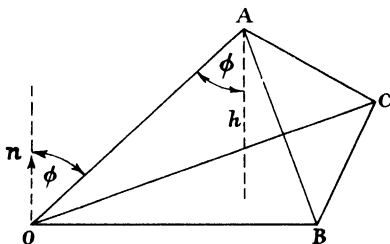


Fig. 182.

Consider now the *triple vector product*  $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$ . This will be a vector perpendicular both to  $\mathbf{a}$  and  $(\mathbf{b} \wedge \mathbf{c})$ , therefore this vector must be in the plane of  $\mathbf{b}$  and  $\mathbf{c}$ . The product may be evaluated by the use successively of (37), (33) and (34), and (25).

$$\begin{aligned} \mathbf{b} \wedge \mathbf{c} &= \Sigma(b_y c_z - b_z c_y)\mathbf{i} \\ \therefore \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \wedge \Sigma(b_y c_z - b_z c_y)\mathbf{i} \\ &= \Sigma\{a_y(b_x c_z - b_z c_x) - a_z(b_x c_y - b_y c_x)\}\mathbf{i} \\ &= (a_x c_x + a_y c_y + a_z c_z)(b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \\ &\quad - (a_x b_x + a_y b_y + a_z c_z)(c_x \mathbf{i} + c_y \mathbf{j} + c_z \mathbf{k}) \\ \text{i.e.} \quad \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ &= \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}), \quad \dots \dots (48) \end{aligned}$$

Note that  $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$  is not equal to  $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$ .

#### 142. Moment of a Force about an Axis.

Let the given axis pass through a point O, and let unit vector in that direction be  $\mathbf{n}$ . Let the force pass through P, where  $\mathbf{OP} = \mathbf{r}$ .

Then, the moment of  $\mathbf{F}$  about the axis is

$$\mathbf{n} \cdot (\mathbf{r} \wedge \mathbf{F}).$$

This expression is independent of the choice of O and P, so long as they are respectively on the axis and the line of action of the force; for

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{O'P'} \wedge \mathbf{F}) &= \mathbf{n} \cdot (\mathbf{O'O} \wedge \mathbf{F}) + \mathbf{n} \cdot (\mathbf{OP} \wedge \mathbf{F}) \\ &\quad + \mathbf{n} \cdot (\mathbf{PP'} \wedge \mathbf{F}) \\ &= \mathbf{n} \cdot (\mathbf{OP} \wedge \mathbf{F}), \end{aligned}$$

since  $\mathbf{O'O}$  is parallel to  $\mathbf{n}$ , and  $\mathbf{PP'}$  is parallel to  $\mathbf{F}$ .

Using this result,

$$\mathbf{n} \cdot (\mathbf{r} \wedge \mathbf{F}) = \mathbf{n} \cdot (\mathbf{MN} \wedge \mathbf{F}),$$

where MN is the common perpendicular to  $\mathbf{n}$  and  $\mathbf{F}$  (Fig. 183). Now  $\mathbf{MN} \wedge \mathbf{F}$  is a vector perpendicular to  $\mathbf{MN}$  and  $\mathbf{F}$  of magnitude  $MN \cdot F$ . But  $\mathbf{F}$  and  $\mathbf{n}$  are also perpendicular to  $\mathbf{MN}$ .

$\therefore \mathbf{F}$ ,  $\mathbf{n}$  and  $\mathbf{MN} \wedge \mathbf{F}$  are coplanar (Fig. 184). If, therefore,  $\theta$  is the angle between  $\mathbf{F}$  and  $\mathbf{n}$

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} \wedge \mathbf{F}) &= \mathbf{n} \cdot (\mathbf{MN} \wedge \mathbf{F}) \\ &= MN \cdot F \sin \theta. \quad \dots (49) \end{aligned}$$

#### 143. Motion of a System of Particles.

Consider a system of particles  $m_1, m_2$ , etc., at  $\mathbf{r}_1, \mathbf{r}_2$ , etc., relative to a fixed point O.

If  $M = \Sigma m$  and  $\bar{\mathbf{r}}$  is the mass centre, then

$$M\bar{\mathbf{r}} = \Sigma m\mathbf{r}, \therefore M\ddot{\bar{\mathbf{r}}} = \Sigma m\ddot{\mathbf{r}}.$$

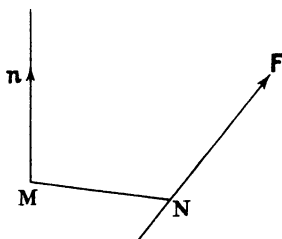


Fig. 183.

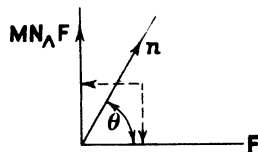


Fig. 184.



But  $m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_1$ , where  $\mathbf{F}_1$  is the resultant force acting on  $m_1$ ,  
 $\therefore M\ddot{\mathbf{r}} = \Sigma \mathbf{F}$ , .. .. (50)

where internal reactions do not appear in  $\Sigma \mathbf{F}$ .

*Hence the motion of the centre of gravity is the same as if the whole mass were concentrated there with the external forces acting on that whole mass parallel to their original directions.*

It follows from (50) that

$$M\ddot{x} = \Sigma X, \quad M\ddot{y} = \Sigma Y, \quad M\ddot{z} = \Sigma Z,$$

where  $\Sigma X$ ,  $\Sigma Y$ ,  $\Sigma Z$  are the components of the external forces.

**144. The Rate of Change of Angular Momentum about a fixed Axis is equal to the Sum of the Moments of the Forces about that Axis.**

Let the direction of the axis be that of  $\mathbf{n}$ , through O, and let one of the forces, say  $\mathbf{F}_1$ , act through  $P_1$ , where  $\mathbf{OP}_1 = \mathbf{r}_1$ . The velocity of  $P_1$  is  $\dot{\mathbf{r}}_1$ . Then :

Angular momentum of system =  $\Sigma m \mathbf{n} \cdot (\mathbf{r} \wedge \dot{\mathbf{r}}) = \mathbf{H}$ , say,

$$\begin{aligned} \therefore \frac{d\mathbf{H}}{dt} &= \Sigma m \mathbf{n} \cdot (\dot{\mathbf{r}} \wedge \dot{\mathbf{r}}) + \Sigma m \mathbf{n} \cdot (\mathbf{r} \wedge \ddot{\mathbf{r}}) \\ &= \Sigma \mathbf{n} \cdot (\mathbf{r} \wedge m\ddot{\mathbf{r}}). \end{aligned}$$

But

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_1,$$

$$\therefore \frac{d\mathbf{H}}{dt} = \Sigma \mathbf{n} \cdot (\mathbf{r} \wedge \mathbf{F}), \quad \dots \dots (51)$$

which proves the proposition.

$$\begin{aligned} \text{Again } \Sigma m \mathbf{n} \cdot (\mathbf{r} \wedge \dot{\mathbf{r}}) &= \Sigma m \mathbf{n} \cdot (\mathbf{OP} \wedge \dot{\mathbf{OP}}) \equiv \mathbf{H} \\ &= \Sigma m \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{OG}}) + \Sigma m \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{GP}}) \\ &\quad + \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{OG}}) + \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{GP}}) \\ &= \Sigma m \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{OG}}) + \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{GP}}) \\ &= M \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{OG}}) + \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{GP}}), \end{aligned}$$

where  $M = \Sigma m$ , since  $\Sigma m \mathbf{GP} = 0$  and therefore  $\Sigma m \dot{\mathbf{GP}} = 0$ .

$$\begin{aligned} \text{But } \Sigma \mathbf{n} \cdot (\mathbf{OP} \wedge \mathbf{F}) &= \Sigma \mathbf{n} \cdot (\mathbf{OG} \wedge \mathbf{F}) + \Sigma \mathbf{n} \cdot (\mathbf{GP} \wedge \mathbf{F}) \\ &= \mathbf{n} \cdot (\mathbf{OG} \wedge \Sigma \mathbf{F}) + \Sigma \mathbf{n} \cdot (\mathbf{GP} \wedge \mathbf{F}). \end{aligned}$$

$$\text{Also, } \frac{d\mathbf{H}}{dt} = M \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{OG}}) + \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{GP}})$$

and

$$\begin{aligned} M \mathbf{n} \cdot (\mathbf{OG} \wedge \dot{\mathbf{OG}}) &= \mathbf{n} \cdot (\mathbf{OG} \wedge \Sigma \mathbf{F}) \\ \therefore \Sigma m \mathbf{n} \cdot (\mathbf{GP} \wedge \dot{\mathbf{GP}}) &= \Sigma \mathbf{n} \cdot (\mathbf{GP} \wedge \mathbf{F}). \quad \dots (52) \end{aligned}$$

This equation gives the angular acceleration of the system about a parallel axis through its centre of mass and is independent of the motion of that centre of mass.

Equations (50) and (52) completely determine the motion of the system of particles.

**Problems XVI.**

1. The magnitudes and directions of two adjacent sides of a regular hexagon are given by  $\mathbf{a}$ ,  $\mathbf{b}$ . Find the vectors representing the other sides taken in order.

2. Prove (i) that the diagonals of a parallelogram bisect one other; (ii) that the diagonals of a parallelepiped meet and bisect one another.

3. Prove that the internal bisectors of the angles of a triangle are concurrent.

4. The position vectors of three points A, B, C are  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ . If  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ ,  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ ,  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ , determine the lengths of AB, BC and CA.

5. The position vectors of three points A, B, C are  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ . Prove that if  $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{0}$ , where  $\lambda + \mu + \nu = 0$ , the points are collinear.

6. Five forces act from one vertex A of a regular hexagon in the directions of the other vertices and proportional to the distances of those vertices from A. Prove that the resultant of the forces is  $6\mathbf{AG}$ , where G is the centroid of the hexagon.

7. If  $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ ,  $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ , find the numerical values of  $\mathbf{a} \cdot \mathbf{b}$  and  $\mathbf{a} \cdot \mathbf{c}$  and verify that

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

Calculate also the value of  $\mathbf{a} \times \mathbf{b}$ .

8. If  $l, m, n$  and  $l', m', n'$  are the direction cosines of  $\mathbf{a}$  and  $\mathbf{b}$  respectively, prove that

$$\sin^2 \theta = (mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2,$$

where  $\mathbf{a}$ ,  $\mathbf{b}$  are inclined at an angle  $\theta$ .

9. In the expression  $\mathbf{r} \wedge \mathbf{F}$  for the moment of a force  $\mathbf{F}$  about the origin O, put  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ ,  $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ , and deduce that the moment of the force about any straight line through O is the resolved part along this line of the vector moment of  $\mathbf{F}$  about O. (N.B. Take the  $x$ -axis as any straight line.)

10. A force  $\mathbf{F}$  represented by the vector  $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$  acts through the point  $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ . Find the torque of the force about the point  $-3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ .

## APPENDIX I

### PRODUCTS OF INERTIA AND PRINCIPAL AXES

If  $m$  is the mass of an element of a plane lamina and  $(x, y)$  the co-ordinates of  $m$  with respect to co-ordinate axes, then  $\Sigma mxy$  is called the *product of inertia* of the lamina with respect to these axes. If the axes are chosen so that  $\Sigma mxy = 0$ , and it will subsequently be proved that this can be done, then these axes are called the *principal axes* of the lamina, and the moments of inertia about these axes are called the *principal moments of inertia*.

To find the product of inertia of a rectilinear lamina about any two given perpendicular axes, a suitable method would be to split the lamina up into triangles, and make use of the fact that a mass  $\frac{M}{3}$ , where  $M$  is the mass of a triangle, placed at each of the three middle points of the sides will be equimomental with the triangle, and will have the same product of inertia about those axes. Consider the right-angled triangular lamina,  $ABC$ , of mass  $M$ , where  $AB = c$ ,  $AC = b$  (Fig. 185).

Using the equivalent system above, the product of inertia for the axes  $AB, AC$  is

$$\frac{M}{3} \frac{b}{2} \frac{c}{2}, \text{ i.e. } M \frac{bc}{12}.$$

For the axes  $AB$  and a perpendicular through  $B$ , the product is

$$\frac{M}{3} \frac{b}{2} c + \frac{M}{3} \frac{b}{2} \frac{c}{2}; \text{ i.e. } M \frac{bc}{4}.$$

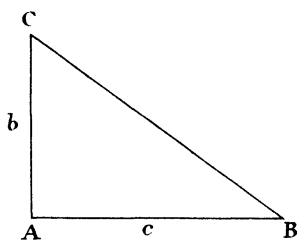


Fig. 185.

To find the product of inertia of the quadrant of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  about the axes, take the element  $PN$  (Fig. 186). If  $m$  is the mass per unit area, the mass of  $PN$  is  $mydx$ , and the distance of its centre of mass from

$OA$  is  $\frac{y}{2}$ ; hence the required product of inertia

$$\begin{aligned} &= m \int_0^a \int_0^b xy \, dy \, dx \\ &= \frac{m}{2} \int_0^a \frac{b^2}{a^2} (a^2 - x^2) x \, dx \\ &= \frac{ma^2b^2}{8} \\ &= M \frac{ab}{2\pi}, \end{aligned}$$

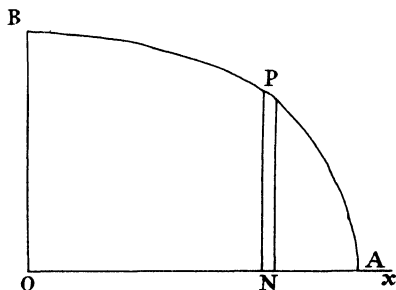


Fig. 186.

where  $M$  is the mass of the quadrant.

If a lamina is symmetrical about an axis, say the  $x$ -axis, then for every element  $m$  on one side of the axis, whose co-ordinates are  $(x, y)$  there is another element of equal mass whose co-ordinates are  $(x, -y)$ . Hence for such a lamina  $\Sigma mxy = 0$ , so that the  $x$ -axis and any line perpendicular to it are principal axes. Thus, if ABCD is a rectangle, and M, N are the middle points of AD, BC respectively, then MN and any line in the plane perpendicular to it are principal axes.

*The product of inertia about any two axes is equal to the product of inertia about any two parallel axes through the centre of mass plus the product of inertia of the whole mass collected at the centre of gravity about the original axes.*

Let G be the centre of mass, and GX, GY axes through G parallel to Ox, Oy respectively (Fig. 187).

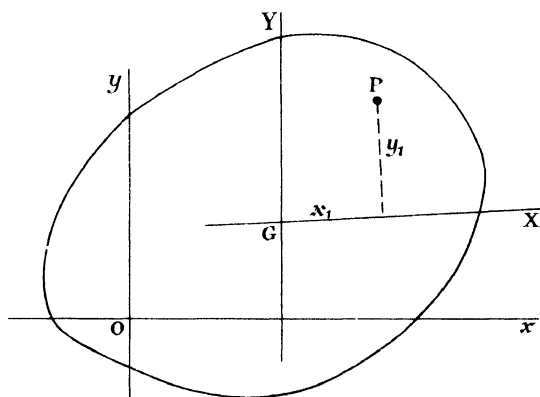


Fig. 187.

Let G have co-ordinates  $\bar{x}, \bar{y}$ , and let  $m$  be an element of the lamina at P, where  $(x_1, y_1)$  are the co-ordinates of P with respect to GX, GY.

The product of inertia with respect to Ox, Oy

$$\begin{aligned} &= \Sigma m(\bar{x} + x_1)(\bar{y} + y_1) \\ &= \Sigma m\bar{x}\bar{y} + \Sigma mx_1y_1 + \Sigma mx_1\bar{y} + \Sigma my_1\bar{x} \\ &= M\bar{x}\bar{y} + \Sigma mx_1y_1 + \bar{y}\Sigma mx_1 + \bar{x}\Sigma my_1, \end{aligned}$$

where M is the mass of the lamina.

But  $\Sigma mx_1 = \Sigma my_1 = 0$ ,  
so that the product of inertia  $= M\bar{x}\bar{y} + \Sigma mx_1y_1$ ,  
which proves the proposition :

If GX, GY are principal axes, then also  $\Sigma mx_1y_1 = 0$  and the product of inertia about Ox, Oy is  $M\bar{x}\bar{y}$ .

As an example, the product of inertia of the rectangular lamina ABCD of mass M about AB, AD is  $M \frac{AB}{2} \frac{AD}{2}$ .

It will now be shown that at *any point of a lamina there can always be found a pair of principal axes.*

Let  $O$  be any point in the lamina and  $m$  the mass of an element at  $P$ , the co-ordinates of which are  $(x, y)$  referred to axes through  $O$  (Fig. 188).  $PN$  is drawn perpendicular to  $ON$ , a line through  $O$  at

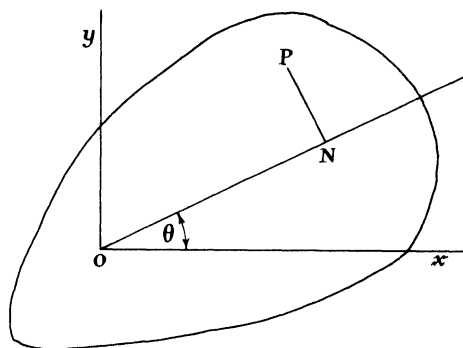


Fig. 188.

angle  $\theta$  to  $Ox$ . Let  $A \equiv \Sigma my^2$ ,  $B \equiv \Sigma mx^2$ ,  $F \equiv \Sigma mxy$ . The product of inertia of the lamina with regard to  $ON$  and a perpendicular to it through  $O$

$$\begin{aligned} &= \Sigma m(ON \cdot PN) = \Sigma m(x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta) \\ &= \Sigma mxy(\cos^2 \theta - \sin^2 \theta) + \Sigma m(y^2 - x^2) \sin \theta \cos \theta \\ &= F \cos 2\theta + \frac{1}{2}(A - B) \sin 2\theta. \end{aligned}$$

If the new axes are to be principal axes,  $\Sigma m(ON \cdot PN) = 0$ ; hence

$$\tan 2\theta = \frac{2F}{B - A};$$

and a value of  $\theta$  can always be found, whatever values  $F$ ,  $B$  and  $A$  may have.

**EXAMPLE.** Find the principal axes at a corner of a rectangular lamina of sides  $a$ ,  $b$ .

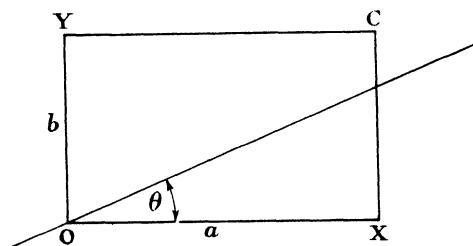


Fig. 189.

Let  $OX = a$ ,  $OY = b$  (Fig. 189) and let  $M$  be the mass of the lamina. Then

$$A = M \frac{b^2}{3}, \quad B = M \frac{a^2}{3}, \quad F = M \frac{ab}{4}.$$

$$\begin{aligned} \therefore \tan 2\theta &= \frac{M \frac{ab}{2}}{\frac{1}{3} M (a^2 - b^2)} \\ &= \frac{3}{2} \frac{ab}{(a^2 - b^2)}. \end{aligned}$$

$\therefore$  the inclinations of the principal axes to  $AB$  are  $\theta$  and  $\frac{\pi}{2} + \theta$ ,

$$\text{where } \theta = \frac{1}{2} \tan^{-1} \frac{3ab}{2(a^2 - b^2)}$$

*If the principal moments of inertia at any point of a lamina are known, the moment of inertia about any other line through the point can be determined, and hence the moment of inertia about any line parallel to it in the plane*

Referring to Fig. 188, where  $O$  is any point in the lamina

$$\begin{aligned} \Sigma mPN^2 &= \Sigma m(y \cos \theta - x \sin \theta)^2 \\ &= \cos^2 \theta \Sigma my^2 + \sin^2 \theta \Sigma mx^2 - 2 \sin \theta \cos \theta \Sigma mxy \\ &= A \cos^2 \theta + B \sin^2 \theta - 2F \sin \theta \cos \theta \end{aligned}$$

But if the axes are *principal axes*,  $F = 0$  hence

$$\Sigma mPN^2 = A \cos^2 \theta + B \sin^2 \theta,$$

and this expression is, therefore, equal to the moment of inertia of the lamina about the axis  $ON$ . [*Corollary* If  $O$  is the centre of gravity, we can obtain the moment of inertia about any line not passing through  $O$  by using the parallel axis theorem.]

**EXAMPLE** Find the moment of inertia of a rectangular lamina  $ABCD$  of mass  $M$  where  $AB = a$ ,  $AD = b$  about a line in its plane through  $D$  parallel to the diagonal  $AC$

Let  $G$  be the centre of mass,  $GX$ ,  $GY$  being principal axes (Fig. 190) Let  $\angle CGX = \theta$ , where

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}}$$

The moment of inertia about  $AC$

$$= A \cos^2 \theta + B \sin^2 \theta$$

$$= M \frac{b^2}{12} \cdot \frac{a^2}{a^2 + b^2}$$

$$+ M \frac{a^2}{12} \cdot \frac{b^2}{a^2 + b^2}$$

$$= \frac{1}{6} M \frac{a^2 b^2}{a^2 + b^2}.$$

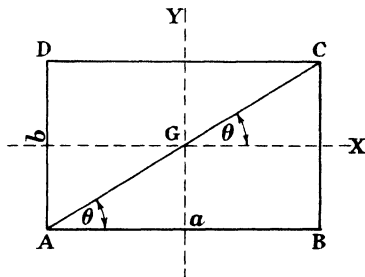


Fig. 190.

Let  $h$  be the length of the perpendicular from D on AC, then

$$h = b \cos \theta = \frac{ab}{\sqrt{a^2 + b^2}}.$$

Then, by the parallel axis theorem, the moment of inertia about a line through D parallel to AC is

$$M \left[ \frac{1}{6} \frac{a^2 b^2}{a^2 + b^2} + \frac{a^2 b^2}{a^2 + b^2} \right] = \frac{7}{6} M \frac{a^2 b^2}{a^2 + b^2}.$$

### *The Momental Ellipse*

Let O be any point in a lamina and Q any other point, such that  $OQ^2$  is inversely proportional to the moment of inertia of the lamina about OQ. It will be proved that the locus of Q is an ellipse, called the *momental ellipse* of the lamina for the point O. We have

$$\Sigma mPN^2 = \frac{M \epsilon^4}{OQ^2}, \text{ where } \epsilon \text{ is a constant.}$$

$$\therefore A \cos^2 \theta + B \sin^2 \theta - 2F \sin \theta \cos \theta = \frac{M \epsilon^4}{OQ^2};$$

or, if  $(x, y)$  are the co-ordinates of Q,

$$Ax^2 + By^2 - 2Fxy = M \epsilon^4.$$

This is the equation of a central conic, and since  $\Sigma mPN^2$  is always positive, OQ is always real and the conic must be an ellipse.

Now we can choose for our arbitrary axes through O (the centre of the conic), the major and minor axes of the ellipse, in which case the term in  $xy$  vanishes. Hence for these axes  $F = 0$ , and therefore the axes of the ellipse through the arbitrary point O must be principal axes of inertia.

Thus through any point in a lamina a pair of principal axes can be found, as has been otherwise proved.

NOTE. When for a body three mutually perpendicular axes are chosen at a point such that  $\Sigma myz$ ,  $\Sigma mzx$  and  $\Sigma mxy$  are all zero, these axes are called the *Principal Axes of Inertia* at the point.

In the case of a lamina the third axis of  $z$  will be perpendicular to the plane, so that for a point in the plane  $\Sigma myz = \Sigma mzx = 0$ . In the case of a body a similar argument to that above leads to the proposition that at every point of a material body three mutually perpendicular principal axes of inertia can be found, these axes being the axes of symmetry of the *momental ellipsoid*.

### *Problems on Products of Inertia and Principal Axes*

1. OABC is a lamina in the shape of a trapezium with OA, CB parallel and OC perpendicular to OA;  $OA = 2a$ ,  $OC = 2a$ ,  $CB = 4a$ . Find the moments of inertia of the lamina about OA and OC. Find also the product of inertia for OA, OC and the angle between OA and one of the principal axes at O.

2. The centre of mass of a plane lamina of mass M is G, and GX, GY are principal axes of inertia at G. Ox, Oy are axes in the plane

of the lamina parallel to GX, GY respectively. If the principal moments of inertia of the lamina are  $Mna^2$  about GX and  $Mm\beta^2$  about GY, and one of the principal axes of inertia at O is inclined at an angle of  $45^\circ$  to Ox, prove that

$$\frac{n - 1}{m - 1} = \frac{\beta^2}{a^2}.$$

3. ABCD is a uniform rectangular lamina whose centre is O and whose mass is M. Four equal particles, each of mass  $\frac{M}{6}$ , are placed

at the mid-points of the sides and a particle of mass  $\frac{M}{3}$  is placed at O.

Show that the moments and product of inertia of the lamina about the axes AB and AD are respectively equal to those of the system of particles about the same axes.

If  $AB = 2AD$  show that one of the principal axes of the lamina at A makes an angle of  $22\frac{1}{2}^\circ$  with AB.

4. ABCD is a plane rectangular lamina, of mass M, in which AB and CD are  $2a$ , and AD and BC are  $2b$  each. Prove that the moment of inertia about the straight line joining A to the mid-point of CD is

$$\frac{8Ma^2b^2}{3(a^2 + 4b^2)}.$$

## APPENDIX II

## INSTANTANEOUS CENTRE OF ROTATION

*Any displacement of a rigid body in a plane is equivalent to rotation about a point in the plane*

Consider any two points A and B of a rigid body (Fig. 191). After displacement, let the line AB be in the position  $A_1B_1$ . Join  $AA_1$ ,  $BB_1$  and let their perpendicular bisectors meet at I. Then  $AI = A_1I$ ,  $BI = B_1I$  and, since the body is rigid,  $AB = A_1B_1$ , from which it follows that the triangles IAB,  $IA_1B_1$

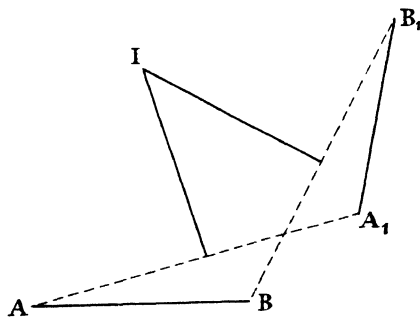


Fig. 191.

are congruent. Hence the triangle IAB can be rotated about I into the position  $IA_1B_1$ . This point I is called the *centre of rotation* for the displacement.

As a plane body continually changes its position, so does the point I and, at any moment, I is called the *Instantaneous Centre of Rotation*.



If the displacement of the body is infinitely small, then  $AA_1$ ,  $BB_1$  above become infinitely small and tangential to the curves traced out by A and B respectively as the body is continually displaced. Therefore the perpendicular bisectors of  $AA_1$ ,  $BB_1$  become the normals to these two loci, and I, the instantaneous centre of rotation, is the point of intersection of these two normals. I will itself describe a curve in the plane, and this curve is called the *space centrode* of I. I will also describe a curve relative to the moving body, and this curve is called the *body centrode* of I.

Consider a rod AB slipping down a wall OB and along a horizontal floor OA (Fig. 192).

The locus of A is the line OA, and the locus of B is the line OB, and therefore the instantaneous centre I is where the perpendiculars to these lines meet.

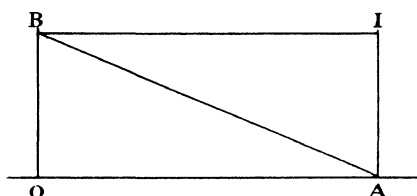


Fig. 192.

Hence BOAI is a rectangle and  $OI = AB$ , a constant. I therefore describes a quadrant of a circle, centre O and radius AB. This quadrant is the space centrode of I. Since angle AIB is a right angle, I describes a semicircle with respect to AB, and this locus is the body centrode of I.

Since at any moment a lamina is moving round I, every point P in the body is instantaneously describing a circle round I and is therefore moving perpendicular to PI.

*To determine the Co-ordinates of the Instantaneous Centre of Rotation*

Let the centre of gravity, G, of the body have velocities  $u$ ,  $v$ , parallel to any fixed axes  $Ox$ ,  $Oy$ , and let  $\omega$  be the angular velocity about an axis perpendicular to the plane  $xOy$  (Fig. 193). Consider any other point P of the body, whose co-ordinates, referred to G as pole are  $(r, \theta)$  or  $(x_1, y_1)$ .

Velocity of P parallel to  $Ox$

$$= u - r\omega \sin \theta = u - \omega y_1.$$

Velocity of P parallel to  $Oy$

$$= v + r\omega \cos \theta = v + \omega x_1.$$

If P is the instantaneous centre, these velocities are separately zero, therefore

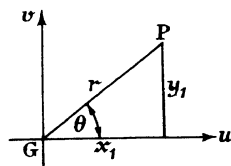


Fig. 193.

$$y_1 = \frac{u}{\omega}; \quad x_1 = -\frac{v}{\omega}.$$

from which the required co-ordinates of P can be obtained.

*Use of the Instantaneous Centre of Rotation*

If  $u$ ,  $v$  are the velocities of the centre of gravity, G, parallel to fixed axes  $Ox$ ,  $Oy$ , and  $\omega$  the angular velocity of the body of mass  $M$ , the equations determining the motion of the body are  $M\ddot{u} = X$ ,

$M\dot{v} = Y$  and  $Mk^2\dot{\omega} = L$ , where  $X$  and  $Y$  are the components of the external forces parallel to the fixed axes,  $L$  the torque of the external forces about  $G$ , and  $k$  the radius of gyration of the body about the perpendicular axis through  $G$ . The effective forces acting at  $G$  are  $M\dot{u}$ ,  $M\dot{v}$  and  $Mk^2\dot{\omega}$  is the effective couple about  $G$ , these three quantities being equivalent to the original forces.

Hence, if  $L_1$  denotes the sum of the moments of the external forces about the instantaneous centre  $P$  (Fig. 194),

$$L_1 = M\dot{u}y_1 - M\dot{v}x_1 + Mk^2\dot{\omega}.$$

But  $y_1 = \frac{u}{\omega}$  and  $x_1 = -\frac{v}{\omega}$ ,

$$\begin{aligned} \therefore L_1 &= M\dot{u} \frac{u}{\omega} + M\dot{v} \frac{v}{\omega} + Mk^2\dot{\omega} \\ &= \frac{M}{\omega} (u\dot{u} + v\dot{v} + k^2\omega\dot{\omega}) \\ &= \frac{M}{2\omega} \frac{d}{dt} (u^2 + v^2 + k^2\omega^2). \end{aligned}$$

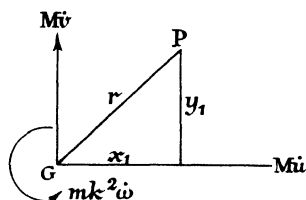


Fig. 194.

Since  $P$  is the instantaneous centre,  $GP$  must be turning round  $P$  with angular velocity  $\omega$ , and therefore the velocity of  $G$  is  $r\omega$  perpendicular to  $GP$ ,

$$\therefore u^2 + v^2 = r^2\omega^2.$$

$$\therefore L = \frac{M}{2\omega} \frac{d}{dt} [(r^2 + k^2)\omega^2]. \quad \dots \quad (1)$$

If  $K$  denotes the radius of gyration about  $P$ ,

$$K^2 = r^2 + k^2.$$

$$\therefore L = \frac{M}{2\omega} \frac{d}{dt} (K^2\omega^2).$$

$K$ , however, being in general a variable,

$$L = MK^2\dot{\omega} + MK\omega \frac{dK}{dt}. \quad \dots \quad (2)$$

If, therefore, there is either a fixed axis through  $G$ , or  $GP$  remains constant in length, i.e. if the instantaneous centre is fixed in the body,

$$L = MK^2\dot{\omega}, \quad \dots \quad (3)$$

an equation similar to that for an axis through  $G$ . If the body starts from rest  $\omega = 0$ , therefore equation (3) still holds and gives the value of the *initial* angular acceleration.

In the case of a *small* oscillation of the body, the term  $MK\omega \frac{dK}{dt}$

may be neglected, since  $\omega$  and  $\frac{dK}{dt}$  are so small that their squares and product may be neglected. Hence, as before,

$$L = MK^2\dot{\omega}.$$

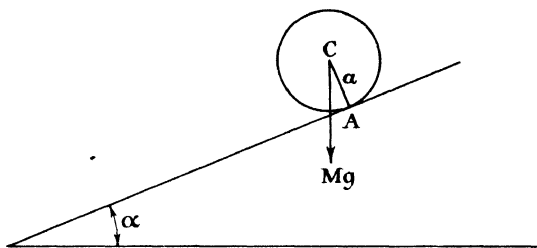


Fig. 195.

In all other cases, equation (2) must be used when taking moments about the instantaneous centre.

Consider the case of a sphere, of mass  $M$  and radius  $a$ , rolling without slipping down an inclined plane of inclination  $\alpha$ .

Let  $A$  be the point of contact of the sphere with the plane (Fig. 195). Now  $A$  is the instantaneous centre and it is at a fixed distance  $a$  from the centre of gravity, hence we can use  $L = MK^2\omega$  and take moments about  $A$ .

$$\text{We therefore have} \quad Mga \sin \alpha = M \left( \frac{2}{5} a^2 + a^2 \right) \dot{\omega}.$$

$$\therefore \quad \dot{\omega} = \frac{5g}{7a} \sin \alpha,$$

which gives the angular acceleration of the sphere.

#### *Problems on the Instantaneous Centre of Rotation*

1. A bar  $AB$  moves in the plane of rectangular axes  $Ox$ ,  $Oy$  so that it always touches the parabola  $y^2 = 4ax$ , and the end  $A$  of the bar moves along the  $x$ -axis. Find the co-ordinates of the instantaneous centre in terms of  $\theta$ , the angle between  $AB$  and  $Oy$ , and prove that the equation of the space centrode is

$$ay^2 + 4x(a-x)^2 = 0.$$

Prove also that, referred to  $A$  as pole and  $AB$  as initial line, the equation of the body centrode is

$$r = 2a \sin \theta \sec^2 \theta.$$

2. Two rods  $AB$  and  $BC$  are fastened at  $B$  so that  $ABC$  is a right angle. The rods are constructed to move in a plane so that  $AB$  touches a fixed circle and  $BC$  passes through a fixed point. Show that the space locus of the instantaneous centre is a circle and that the body locus is a circle with twice the radius of the space locus.

3. The end  $A$  of a given rod is compelled to move on the  $Oy$  axis, whilst the rod itself always passes through a fixed point  $B$  on  $Ox$ , where  $OB = a$ . Show that the locus in space of the instantaneous centre is the parabola  $y^2 = a(x-a)$ .

4. A plane lamina moves in any manner in its own plane. The

velocity of a point P fixed in the lamina relative to rectangular axes  $Ox, Oy$  fixed in the plane is  $(u, v)$ , and the angular velocity of the lamina about an axis perpendicular to its plane is  $\omega$ .  $PX, PY$  are co-ordinate axes parallel to  $Ox, Oy$ , respectively. If the co-ordinates of the instantaneous centre of acceleration J are  $(X, Y)$  relative to  $PX, PY$ , prove that

$$\dot{u} - \omega^2 X - \dot{\omega} Y = 0, \quad \dot{v} - \omega^2 Y + \dot{\omega} X = 0.$$

If the lamina is a circular disc of radius  $r$ , which rolls without slipping along  $Ox$  and whose centre has a constant acceleration  $2f$ , prove that J traces out the circle

$$X^2 + Y^2 - rY = 0.$$

## Miscellaneous Problems.

1. Part of the surface of a sphere is cut off by a plane, and from the centre  $O$  of the sphere a straight line  $ODC$  is drawn cutting the plane at right angles in  $D$  and the surface in  $C$ . Prove that the centre of gravity of the part of the surface cut off will divide  $CO$  in the ratio  $2 : 1$ , if the density at any point  $P$  of the surface varies as  $1 - \cos \widehat{COP}$ .

2. If a circular disc of radius  $a$ , whose thickness varies as  $\sqrt{1 - \frac{r^2}{a^2}}$  where  $r$  denotes distance from the centre, be bisected by a diameter, prove that the centre of gravity of either half is  $\frac{3}{8}a$  from the centre.

3. The middle points of opposite sides of a jointed quadrilateral are connected by light rods of length  $l, l'$ . If  $T, T'$  be the tensions in these rods, prove that  $\frac{T}{l} + \frac{T'}{l'} = 0$ .

4. A uniform rod of length  $a$  and specific gravity  $s$  can turn about one end which is fixed at a height  $h$  above the surface of water into which the other end of the rod dips. Prove that if  $s < 1$  and  $a$  is less than a certain value, the vertical position is the only position of equilibrium; but if  $a$  exceeds a certain value, the vertical position is one of unstable equilibrium and the rod will be in a position of stable equilibrium when it is inclined to the vertical.

5. Water is poured into a vessel of any shape. Prove that at the instant when the centre of gravity of the vessel and the contained water is lowest it is at the level of the water surface.

6. A uniform india-rubber cord has a length of 26 inches under a tension of  $2\frac{1}{2}$  pounds weight, and a length of 20 inches under a tension of one pound weight; calculate the amount of work done in stretching it from its natural length to a length of 30 inches, and draw a work diagram.

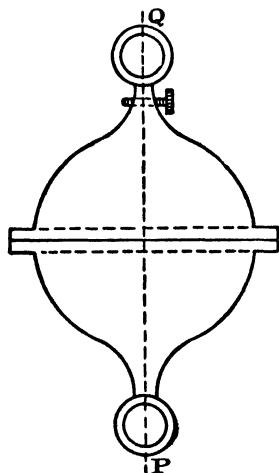


Fig. 196.

7. A light elastic string of natural length  $l$  has one extremity fixed at a point  $A$  and the other attached to a stone, the weight of which in equilibrium would extend the string to a length  $l'$ . Prove that, if the stone is dropped from rest at  $A$ , it will come to instantaneous rest at a depth  $\sqrt{(l'^2 - l^2)}$  below the equilibrium position.

8. Two flanged hemispheres fit closely together and are airtight. They are exhausted to  $\frac{1}{10}$  atmosphere through a hole in one handle. The tap is closed and a handle screwed on. Find the pull in direction  $PQ$  which will separate the hemispheres. Radius = 8 cm. (neglect thickness), Barometer = 76 cm., Specific Gravity of mercury = 13.6.

9. A mass hangs vertically by a light elastic cord, whose length was increased by the weight of the mass from 15 ins. to 17 ins. The mass then receives an impulse vertically downwards, giving it the velocity it would gain by falling 6 ins. freely under gravity. How far will it descend before coming to momentary rest, and how far will it then ascend above the position of equilibrium?

10. A uniform chain 6 feet long, having a mass of 2 lbs. per foot is laid in a straight line along a rough horizontal table for which the coefficient of friction is  $\frac{1}{2}$ , a portion hanging over the edge of the table so that slipping is just about to occur. If a slight disturbance sets the chain slipping, find the tension at the edge of the table when  $x$  feet have slipped off and the velocity then taken.

11. Two beads connected by a string are held at rest on a vertical circular wire with the string horizontal, and above the centre. Their masses are  $m, m'$ , and the string subtends an angle  $2\alpha$  at the centre. If the beads are released, show that the tension of the string when it makes an angle  $\theta$  with the horizontal is

$$\frac{2mm'g \tan \alpha \cos \theta}{m + m'}.$$

12. A mass is suspended by a light elastic string from a point A and produces an extension  $c$ , the natural length being  $a$ . Show that if the mass be raised through a vertical distance less than  $c$  and be then let go from rest, it makes oscillations of the same period as a simple pendulum of length  $c$ .

If the mass is raised up to A and let fall, show that the maximum extension of the string is equal to  $c(1 + \sec \alpha)$ , where  $\alpha$  is the acute angle given by  $\tan^2 \alpha = 2a/c$ ; and that this maximum extension is attained at a time  $(\pi - \alpha)\sqrt{c/g}$  after the string first becomes taut.

13. A rod of length  $2l$  with one end on a horizontal plane leans against a circular cylinder of radius  $a$  whose axis lies in the plane,  $\lambda$  and  $\lambda'$  being the angles of friction at the two contacts. If  $\theta$  be the greatest angle the rod can make with the horizontal in such a position, show that

$$l \sin \theta \sin (\theta + \lambda - \lambda') = a \sin \lambda \cos \lambda'$$

provided this value of  $\theta$  is greater than  $\tan^{-1} \frac{a}{2l}$ .

14. A weightless rod of length 3 feet, with equal heavy rings at the ends, one of which can slide on a smooth horizontal wire, is describing small oscillations in the vertical plane containing the wire. Show that the period of oscillation is about 1.36 seconds.

15. A regular octahedron formed of twelve equal rods of weight  $w$  freely jointed is suspended from one corner. Prove that the thrust in each horizontal rod is  $3w/\sqrt{2}$ .

16. A particle of mass  $m$  moving with velocity  $u$  impinges on a particle of mass  $M$ . If after the impact the component velocities of the mass  $m$  are  $u'$  and  $v'$  in directions along and perpendicular to

the original direction of motion, find the component velocities of the mass  $M$  in these directions.

If there is no loss of energy in the impact prove that the greatest value that  $v'$  can have is  $\frac{M}{M+m} u$ , and that this occurs when the

line of the impulse makes an angle of  $\frac{\pi}{4}$  with the original direction of motion.

17. A bullet of mass  $m$  is fired into a block of wood of mass  $M$ , which is free to move on a smooth horizontal table and penetrates it to a depth  $a$ . Show that, at the instant when the bullet comes to rest relative to the block, the block has moved through a distance  $ma/(M+m)$ , the stress between the bullet and the block being assumed constant, so long as there is any relative motion.

18. A train consists of an engine of mass  $M$  tons, and two wagons, each of mass  $m$  tons. At the start the buffers are in contact, and when the coupling chains are tight the buffers are  $a$  feet apart. The train starts with the engine exerting a constant tractive force of  $F$  tons weight. Neglecting friction, prove that the second wagon starts with velocity  $v$  feet per second, where

$$v^2 = \frac{2Fga(2M+m)}{(M+2m)^2}.$$

19. A heavy string of variable density is hung up from two points. Prove that if  $T_1, T_2, T_3$  are the tensions at  $A, B, C$  where the inclinations of the tangents to the horizon at  $a - \beta, a, a + \beta$ , and  $w_1, w_2$  the weights of the parts of the string  $AB, BC$  respectively,

then  $\frac{1}{T_1} + \frac{1}{T_3} = \frac{2 \cos \beta}{T_2}$  and  $\frac{T_1}{w_1} = \frac{T_3}{w_2}$ .

20. Three particles are projected simultaneously and in the same vertical plane from a point with velocities  $v_1, v_2, v_3$  in directions making angles  $\alpha_1, \alpha_2, \alpha_3$  with the horizontal. Show that the area of the triangle formed by the particles at any time is proportional to the square of the time elapsed from the instant of projection, and that the three particles will always lie on a straight line if

$$\sum v_2 v_3 \sin(\alpha_2 - \alpha_3) = 0.$$

21. A heavy particle is fastened to a point of a uniform elastic thread which is stretched between two fixed points  $A, B$  in the same vertical line. The particle is let go from rest in such a position that the tensions in the two parts of the thread are initially equal; show that, if the tension is sufficiently great, the oscillation will have the same period as a simple pendulum of length  $c$ , where  $2c$  is the vertical distance through which the particle falls before next coming to rest. Express the necessary limit of the initial tension in terms of the weight of the particle and the unstretched lengths  $a, b$ , of the two parts of the thread.

22. Two particles each of mass  $m$  are connected by a light inextensible string which passes through a hole in a horizontal table. One particle hangs freely while the other can slide in a smooth

elliptic groove cut on the table, the hole being at one focus. The latter particle is initially at the end of the major axis remote from the hole and is slightly displaced. Prove that, as it passes through the other extremity of the major axis, in the subsequent motion, the horizontal pressure on the groove is  $mg(1 - 3e)/(1 + e)$  where  $e$  is the eccentricity of the ellipse.

23. An inelastic particle of mass  $m$  is moving on a smooth horizontal plane with velocity  $V$  and strikes the middle point of the base of a smooth wedge of mass  $M$  perpendicularly. Prove that the wedge begins to move with velocity  $mV \sin^2 \alpha / (M + m \sin^2 \alpha)$  where  $\alpha$  is the angle of the wedge.

24. Two particles of mass  $m$  and  $M$  respectively are connected by a weightless inextensible string of length  $2l$ , which passes through a small smooth hole in a smooth horizontal table on which  $M$  moves, while  $m$  hangs vertically. Initially  $M$  is at a distance  $l$  from the hole and it is projected with velocity  $V$  at right angles to the string. Show that, if  $3MV^2 > 8lmg$ ,  $m$  will be raised up to the hole.

25. A chain is formed of  $n$  equal particles attached at regular intervals to a light string. If the chain is heaped on a table and lifted off the table by a vertical force equal to its weight, show that the energy that has been dissipated when the chain is clear of the table is  $\frac{1}{3} \left( 1 - \frac{1}{2n} \right)$  times the work done by the force.

26. A fine smooth thread is passed round two pegs  $S_1$  and  $S_2$  on a smooth horizontal plane and a small bead (mass  $m$ ) slides on the thread. The bead is placed at  $P$  ( $S_1P = S_2P$ ) and struck a blow parallel to  $S_1S_2$  which gives it initial speed  $u$ . Find the initial tension of the thread and determine the motion which ensues.  $S_1S_2 = 2l$ , length of thread  $= 2a + 2l$ . Also find the tension of the string when the mass is in the same straight line as  $S_1S_2$ .

27. One elliptic cylinder rests on another equal one, the lengths being horizontal, and the minor axes of the cross sections in the same vertical plane. Prove that for stability the eccentricity of the cross sections must exceed  $1/\sqrt{2}$ .

28. Three particles, of mass  $m, m', m''$  are attached to the points  $A, B, C$  of an inextensible string. They are laid on a smooth horizontal table with the portions of string between the particles taut, and the angle  $ABC$  obtuse, and equal to  $(\pi - \alpha)$ . A blow  $P$  is applied to  $C$  parallel to and in the direction of  $AB$ . Prove that  $m$  begins to move with velocity

$$m'P \cos^2 \alpha / [m'(m + m' + m'') + mm'' \sin^2 \alpha].$$

29. A uniform rod of length  $2a$  is moving in a vertical plane with horizontal velocity  $U$ , and makes an angle  $\alpha$  with the vertical in the direction opposite to that of its horizontal velocity. Prove that if it strikes the ground, supposed rough, with a vertical velocity  $V$  it will not attain a vertical position unless

$$U > V \tan \alpha + 4(\frac{1}{2}ag)^{\frac{1}{2}} \sin \frac{1}{2}\alpha / \cos \alpha.$$



30. A heavy uniform plank of length  $2l$  rests symmetrically across a rough beam whose section is a semicircle of radius  $a$ , and has a weight  $W$  at each end; it is then carefully tilted without slipping through an angle  $\theta$ . Show that it will rest in this position if a force  $P$  be applied at the lower end perpendicular to the plank, provided that  $(l - a\theta) \sin \theta$  is not greater than  $\mu l \cos \theta$ , where  $\mu$  is the coefficient of friction.

31. A bullet of mass  $m$  is fired from a rifle of mass  $M$  which is suspended in a horizontal position so that it can turn freely round a horizontal axis of suspension. The rifle, after being fired, rotates through an angle  $\alpha$  under the action of gravity. Show that the velocity of the bullet on leaving the rifle is  $2\frac{M}{m}\sqrt{\frac{g}{h}}K \sin \frac{\alpha}{2}$ , where  $h$

is the distance of the centre of gravity of the rifle (supposed to lie in the axial line of the barrel) from the axis of suspension and  $K$  is the radius of gyration of the rifle about the axis of suspension.

32. A uniform lamina, whose shape is a square, is rotating in its plane about one angular point; a second adjacent angular point is then seized and the first released. Prove that the angular velocity is reduced to one-fourth of its former value.

33. A uniform rod  $AB$  is free to turn about its end  $A$ ; find the points of its length at which a blow given to it perpendicular to its length will cause impulsive actions at  $A$  equal in each case to  $\frac{1}{n}$ th of that of the blow.

Deduce the position of the Centre of Percussion of the rod.

34. A smooth wedge of mass ( $M$ ) and angle ( $\alpha$ ) rests on a smooth horizontal table on which it is free to slide. Another wedge of the same mass and angle is placed on the face of the first so that the upper face of the second is horizontal. A smooth particle of mass ( $m$ ) is placed on the upper face of the second, and the whole system is allowed to move freely under gravity. Prove that if the two wedges can only move in directions perpendicular to their edges, ( $m$ ) descends vertically with acceleration

$$\frac{2(m + M) \sin^2 \alpha}{M + (2m + M) \sin^2 \alpha} g.$$

35. A cube oscillates about one edge which is fixed and in a horizontal position. Prove that the length of the simple equivalent pendulum is  $\frac{2}{3}\sqrt{2} \cdot a$  where  $a$  is the length of the edges of the cube. Also prove that when the cube oscillates about a diagonal of a face, the diagonal being horizontal, the length of the simple equivalent pendulum is  $\frac{5a}{6}$ .

36. Two particles of mass  $M$  and  $m$  ( $M > m$ ) are placed on the smooth faces of a light wedge which rests on a smooth horizontal plane. The faces of the wedge are inclined to the horizontal at

angles  $\alpha$  and  $\beta$  respectively. If the system starts from rest, show that the smaller particle will move up the face on which it is placed if

$$\tan \beta < \frac{M \sin \alpha \cos \alpha}{M \sin^2 \alpha + m}.$$

37. A uniform rod of length  $2l$  and mass  $m$  is hinged at one end and lies on a horizontal table. It is struck a blow  $P$  at the other end at right angles to its length. Find the initial impulse at the hinge and the initial angular velocity. If the coefficient of friction of the table be  $\mu$ , through what angle will the rod rotate before it comes to rest?

38. One end of a string is attached to a fixed point  $O$ , and particles of masses  $m, m'$  are attached to the string at intervals at the points  $A, A'$  respectively. The system is held at rest with the points  $O, A, A'$  in a vertical plane and the strings  $OA, AA'$  taut. It is now released from rest. Prove that the tension in the string  $OA$  is initially

$$\frac{m(m + m')g \cos \alpha}{m + m' \sin^2 \beta}$$

where  $\alpha$  is the angle  $OA$  makes with the vertical and  $\beta$  is the angle between  $OA$  and  $AA'$ .

39. Two equal particles connected by an elastic string which is just taut lie on a smooth table, the string being such that the weight of either particle would produce in it an extension  $a$ . Prove that, if one particle is projected with velocity  $u$  directly away from the other, each will have travelled a distance  $u\pi\sqrt{(a/8g)}$  when the string first returns to its natural length.

40. A particle of mass  $m$  is placed on the smooth slant surface of a wedge of mass  $M$  and angle  $\alpha$ . The wedge moves (edge foremost) along a smooth horizontal plane under the action of a constant horizontal force  $P$  perpendicular to its edge, all the motion being parallel to the same vertical plane. Prove that the acceleration of the wedge is

$$\frac{P - mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

41. A hollow cylinder (of radii  $a, b$ ) has one end pressed against a rough plane with a pressure  $P$ . Assuming  $P$  to be uniformly distributed over the area of contact, prove that the couple necessary to rotate the cylinder about its axis is equal to  $\frac{3}{8}\mu P(a^2 + ab + b^2)/(a + b)$ .

42. The mean distance of Mars from the Sun being 1.524 times that of the Earth, find the time of revolution of Mars about the Sun.

43. An equilateral triangular lamina is rotating about its own plane about its centroid, and one corner is suddenly fixed. Prove that the angular velocity is immediately reduced to one-fifth of its original amount, and find the loss of kinetic energy.

44. A man throws a stone with velocity  $u$  at elevation  $\alpha$ , and  $t$  seconds later he throws another stone with velocity  $v$  at elevation  $\beta$ , so as to hit the first stone. Prove that

$$2uv \sin (\alpha - \beta) = (u \cos \alpha + v \cos \beta)gt.$$

45. A cone of semi-vertical angle  $\alpha$  is composed of material the density of which at any point is proportional to the perpendicular distance of that point from the base of the cone. If the cone is placed with its base upon an inclined plane, and is on the point of sliding and toppling over at the same time, prove that the coefficient of friction between the cone and the plane is  $\frac{2}{3} \tan \alpha$ .

46. A mass  $m$  rests on a rough horizontal plane, coefficient of friction  $\mu$ . Find the least force that applied to the mass will just cause it to move and prove that the angle of its line of action makes with the plane an angle  $\tan^{-1} \mu$ . Find the least force when the mass rests on a rough plane inclined at angle  $\alpha$  with the horizontal ( $\alpha < \tan^{-1} \mu$ ).

47. A conical plug is pressed with pressure  $P$  into a conical socket, the radii of the ends of the plug being  $a, b$ . Assuming only the curved surfaces are in contact and the pressure uniformly distributed, prove that the necessary couple to produce rotation is equal to  $\frac{2}{3} \mu P \operatorname{cosec} \alpha (a^2 + ab + b^2) / (a + b)$  where  $2\alpha$  is the angle of the cones.

48. Prove that the directions of motion of all points on a given straight line are tangents to a parabola.

49. Assuming that the effective horse-power of a motor-car of weight  $W$  lb. is  $K(\omega - \beta\omega^2)$ , where  $\omega$  is the angular velocity of the crank shaft, and that the speed of the car is  $k\omega$ , where  $k$  is a constant characteristic of the gear in use, show that in ascending a hill of gradient  $\alpha$  against a resistance  $r$  lb. wt., the speed of the car is given by  $\beta v = k \left( 1 - \frac{W \sin \alpha + r}{550} \cdot \frac{k}{K} \right)$  and hence that the best gear to use is that for which  $k$  is nearest to  $275K / (W \sin \alpha + r)$ .

50. A fly-wheel weighing  $W$  lbs. is carried on an axis of  $a$  feet radius. A light string is wrapped round the axis, and to the other end of the string is attached a weight of  $W'$  lb. The friction is such that if  $W'$  were equal to  $W''$ , the fly-wheel would just be on the point of motion. Starting from rest, the weight  $W'$ , making the fly-wheel rotate, falls  $h$  feet in  $t$  seconds. Assuming that the mass of the flywheel is concentrated at the circumference, show that its radius is

$$a \sqrt{\frac{(W' - W'')gt^2}{2Wb} - \frac{W'}{W}} \text{ feet.}$$

51. A smooth elliptical wire of axes  $a, b$  rotates with constant angular velocity  $\omega$  about the axis  $a$ , which is vertical. Show that a bead free to slide on the wire will remain in equilibrium relative to the wire at a depth  $a^2 g / b^2 \omega^2$  below the centre.

52. A uniform rod of length  $c$  slides between two fixed guides inclined at an angle of  $45^\circ$ , the motion being in one plane. When the angular velocity is  $\omega$ , show that the Kinetic Energy is

$$\frac{1}{2} M \omega^2 \{4c^2 - 3pc\},$$

where  $M$  is the mass of the rod,  $p$  its distance from the intersection of the guides.

53. A telegraph wire is stretched between two poles at a distance ( $a$ ) apart. Prove that the sag of the wire half way between the poles is approximately

$$\frac{1}{8} \frac{Wa}{T} \left( 1 + \frac{5}{48} \frac{W^2}{T^2} \right)$$

where  $W$  is the weight of the wire and  $T$  the tension at the posts.

54. Two equal uniform rods  $AB$ ,  $BC$  are rigidly connected at right angles at  $B$ , and are rotating in their own plane with angular velocity  $\omega$  about  $A$ . The end  $C$  impinges on a fixed smooth inelastic rod at right angles to  $BC$ ,  $A$  being set free; show that after the impact the angular velocity of the body is  $\omega/13$ , and that the velocity of  $C$  along the rod is  $4/13$ ths of its previous velocity in that direction.

55. A shop sign consists of a rectangle  $ABCD$  which can turn freely about its side  $AB$  which is horizontal. The wind blows horizontally with a steady velocity  $v$  and the sign is at rest inclined at an angle  $\alpha$  to the vertical; assuming the wind-thrust on each element of the sign to be  $k$  times the relative normal velocity, find the value of  $\alpha$ , and show that the time of a small oscillation about the position of equilibrium is

$$2\pi \sqrt{\frac{a}{g} \cdot \frac{4v^2 \cos^2 \alpha}{3v^2 \cos \alpha - ga \sin^2 \alpha}}$$

where  $BC = 2a$ .

56. A uniform solid, of specific gravity  $0.75$ , is in the shape of a right circular cylinder surmounted by a hemisphere, the plane base of which coincides with an end of the cylinder. The body floats in water with the hemisphere only above the surface of the water. Find the position of the metacentre and determine whether the equilibrium is stable or unstable.

57. A completely enclosed long box, made of uniform material so that its centre of gravity is at its geometrical centre, has a square cross-section whose side is of length  $2a$ . The box is placed in water so that it floats in water with four of the edges of length  $2a$  vertical and immersed to a depth  $2d$ . Show that if  $(3 - \sqrt{3})a < 6d < (3 + \sqrt{3})a$  then the equilibrium is unstable for angular displacements about the long axis.

Show that the box will float in stable equilibrium with the edges of length  $2a$  inclined to the horizontal at an angle of  $45^\circ$  if half the volume is immersed.

58. The co-ordinates of the vertices  $A$ ,  $B$ ,  $C$ ,  $D$  of a plane quadrilateral are respectively  $(0, 0)$ ,  $(2, 1)$ ,  $(3, 2)$  and  $(-1, 4)$ . Forces  $\lambda \vec{AB}$ ,  $2\lambda \vec{BC}$ ,  $3\lambda \vec{CD}$ ,  $4\lambda \vec{DA}$  act along  $AB$ ,  $BC$ ,  $CD$ ,  $DA$  in the senses indicated by the arrows. Find the magnitude of their resultant and the equation of its line of action.

## PROBLEMS I

**1. 27 ft./sec. ; 46 ft.**

$$2. s = 20 \log_e \left( 1 + \frac{t}{5} \right); t = 5 \left( e^{\frac{s}{20}} - 1 \right) : 5.753 \text{ ml.}; 3.243 \text{ hr.}$$

**3.** 3 ft./sec. ; 12 ft./sec.<sup>2</sup> : 3.509 ft. ; — 28.509 ft.

5.  $\frac{\pi}{6\sqrt{k}}$       6. (1) 10 hr.  
(2) 10.494 ml.; 1.952 ml. per hr.

7. 1019.3 ft.                      8.  $5\frac{9}{11}$  ml. per hr.

9.  $\frac{25\pi}{2}$  ft./sec.;  $1000\left[\frac{\pi}{4} - \frac{1}{3}\right]$  ft.      10. 3.7 ml.; 15 ml./hr.

**11.** 44.7 ft./sec. **12.** 6 ft. ; 7 ft./sec. ; 8 ft./sec.<sup>2</sup>

**13.** 10.38 ft./sec. ; .322 ft./sec.<sup>2</sup>      **14.** 11.8 min.

**15.** 6.697 ft.; .9 ft./sec.      **16.**  $\frac{2\sqrt{35}}{35}$  ft./sec.

$$17. \frac{1}{\sqrt{\mu}} \log_e \frac{\sqrt{\mu}x + \sqrt{\mu x^2 + V^2}}{V}; \frac{V}{\sqrt{\mu}} \sinh \sqrt{\mu}t; \frac{1}{\sqrt{\mu}} \cosh^{-1} 2$$

18.  $\frac{1}{q} \left[ \tan^{-1} \frac{q}{p} + n\pi \right]$  where  $n$  is an integer.

**19.**  $\frac{33.098}{x^2}$  in./sec.      **23.**  $\frac{2\sqrt{3}\pi}{b}$

**2. 54·14.**

**3. 3 in. more.**

**6. 4 in.**

7.  $2\frac{1}{4}$  in. ; 4.557 ft./sec.

**11. 35.1 ft./sec.**

**12.**  $\frac{5\pi^2}{16g}$ .

16.  $\sqrt{\frac{10}{g}} \log_e (10 + 3\sqrt{11})$ . 20. 105 sec.; 73½ ft./sec.; 7½ H.P.

24.  $\frac{4928}{g} \log_e \frac{P - RM}{P - RM - \frac{2}{3} MV}$  sec. where the units are those as given in the latter part of the question : infinite time.

25. 48.05 ml./hr.;  $\frac{24640}{g} \log_e 2$  sec.

26.  $1\frac{1}{2}$  ft./sec.<sup>2</sup>; 648.6 H.P.;  $1303\frac{7}{11}$  H.P.

30. 1.68 ft./sec.<sup>2</sup>;  $45\frac{5}{8}$  ft./sec.

31.  $\frac{1000}{g} \log_e 2$  sec.; 40 ft./sec.

33. 2613.8 H.P.

34.  $2\pi \sqrt{\frac{a^3}{km}}$ .

## PROBLEMS III

1.  $128y = (x - \frac{2}{3}y)^3$ ;  $\frac{1}{2} \int_0^2 (13t^4 + 32t^3 + 64)^{\frac{1}{2}} dt$ .

2.  $\frac{\sqrt{2}y}{V \sin a} = \sin\left(\frac{\sqrt{2}x}{V \cos a}\right)$ ;  $\sqrt{2}\pi V \cos a$ .

3. 33 min. 53 sec. past 10.0 a.m.; 1 ml. 721 yd.; 17.51 ml./hr.

4.  $\omega \sqrt{2a^2 - b^2}$ ;  $\tan^{-1} \frac{a}{\sqrt{a^2 - b^2}}$  with tube;  $\frac{1}{\omega} \log_e \frac{a + \sqrt{a^2 - b^2}}{b}$ .

5.  $\frac{2a(a+b)}{b} \omega \cos \frac{\omega t}{2}$ ,  $\frac{\pi}{2} + \frac{2a+b}{2b} \omega t$ :

$$\frac{a(a+b)}{b} \omega^2 \left[ 1 + \frac{4a(a+b)}{b^2} \cos^2 \frac{\omega t}{2} \right]^{\frac{1}{2}},$$

$$\pi + \tan^{-1} \frac{a \sin \frac{a}{b} \omega t + (a+b) \sin \frac{a+b}{b} \omega t}{a \cos \frac{a}{b} \omega t + (a+b) \cos \frac{a+b}{b} \omega t}.$$

6. Constant —  $\frac{uV^2t}{a^2}$ ;  $\frac{2uV}{a}$ .

7.  $7\frac{1}{2}$  ft./sec.;  $1\frac{7}{8}$  ft./sec.<sup>2</sup>; angles  $\frac{t}{4}$ ,  $\frac{\pi}{2} + \frac{t}{4}$  with initial direction.

8.  $x^2 + y^2 = 2ay$ .

9. (i.)  $\frac{\pi}{8}$  rad. per sec.; (ii.)  $\pi$  in./sec.; (iii.)  $2\pi \sin \frac{3\pi}{16} t$  in./sec.;

$$\frac{5\pi}{16} t \text{ rad.}$$

11.  $\frac{15\sqrt{\pi^2+4}}{8}$ ,  $\tan^{-1}\left(\frac{2}{\pi}\right)$ ;  $\frac{15}{4}$ , 0;  $\frac{15\sqrt{\pi^2+4}}{8}$ ,  $\tan^{-1}\left(-\frac{2}{\pi}\right)$ ;

$$\frac{15\sqrt{\pi^2+1}}{4}, \tan^{-1}\pi.$$

13.  $-a\omega \sin \theta \left[ 1 + \frac{a \cos \theta}{(b^2 - a^2 \sin^2 \theta)^{\frac{1}{2}}} \right],$   
 $-a\omega^2 \left[ \cos \theta + \frac{a(b^2 \cos 2\theta + a^2 \sin^4 \theta)}{(b^2 - a^2 \sin^2 \theta)^{\frac{3}{2}}} \right].$
15.  $\left( 5 - t \right) \left( 1 + 2 \sin \frac{2t}{5} \right) + 5 \cos \frac{2t}{5} = 0; 4 \text{ hr. } 30 \text{ min.}$
16.  $\frac{5\pi}{27} \text{ cm./sec.}; \frac{5\sqrt{3}\pi^2}{1458} \text{ cm./sec.}^2$
17.  $2\omega \sin \theta \left[ 1 - \frac{\cos \theta}{(15 + \cos^2 \theta)^{\frac{1}{2}}} \right];$   
 $2\omega^2 \left[ \cos \theta - \frac{15 \cos 2\theta + \cos^4 \theta}{(15 + \cos^2 \theta)^{\frac{3}{2}}} \right].$
21. (i.) Line through A perpendicular to AX.  
 (ii.)  $\sqrt{bt(4a - bt)}$  cm.  
 (iii.)  $\frac{(2a - bt)\sqrt{b}}{\sqrt{t(4a - bt)}} \text{ cm./sec.}, - \frac{4a^2\sqrt{b}}{(4at - bt^2)^{\frac{3}{2}}} \text{ cm./sec.}^2:$   
 $\frac{\sqrt{b}}{\sqrt{t(4a - bt)}} \text{ rad./sec.}, - \frac{(2a - bt)\sqrt{b}}{(4at - bt^2)^{\frac{3}{2}}} \text{ rad./sec.}^2 \quad \text{where}$   
 OA = a cm.
22.  $\omega' = \frac{AQ}{AP}\omega.$                       24. 11.46 degrees per min.
26.  $\frac{\pi}{249660} \left[ 1049 - \frac{261,261}{2069 - 1900 \cos \theta} \right] \text{ rad. per day, where}$   
 $\theta = \frac{638\pi t}{249660} \text{ rad.}$
28.  $\theta.$                       31.  $\frac{1}{\omega} \log (4 + \sqrt{15}); \tan^{-1} \frac{4\sqrt{15}}{15}.$

## PROBLEMS IV

3. 13,519 ft.; 175.5 ft./sec.  
 4.  $R^2 \tan^2 \alpha - 4h'R \tan \alpha + R^2 - 4hh' = 0.$   
 5.  $\frac{\pi u^4}{43560g^2 \cos^3 \alpha}.$                       10.  $4he(1 + e) \sin \alpha.$   
 12.  $mg \left( 2 - \frac{\pi}{2} \right).$   
 31. (i.)  $\frac{21250g}{(25 + 4x^2)^{\frac{3}{2}}} \text{ dynes};$     (ii.)  $2g(100 - x^2) \text{ ergs};$

- (iii.)  $\frac{2gx}{(25 + 4x^2)^{\frac{1}{2}}} \text{ cm./sec.}^2$
- (i.)  $\frac{34\sqrt{5}g}{5} \text{ dynes ;}$  (ii.) 150g ergs ;
- (iii.)  $\frac{2\sqrt{5}g}{5} \text{ cm./sec.}^2$  ( $x$  measured in cm.)
34.  $-\frac{k}{(bx - x^2)^{\frac{3}{2}}}$  where  $k$  is a constant.
35.  $-\frac{kxy}{(x^2 + y^2)^3}$  where  $k$  is a constant.
42.  $2\sqrt{\frac{g \sin \theta}{1(\cos \theta)}}$

## PROBLEMS V

1.  $\bar{x} = \frac{3}{8}a, \bar{y} = \frac{3}{4}a.$  2.  $\bar{x} = \frac{3}{8}a.$
3.  $\frac{3(b+c)(2a^2 - b^2 - c^2)}{4(3a^2 - b^2 - bc - c^2)}$  from centre of sphere.
4.  $\bar{x} = c [\log_e(1 + \sqrt{2}) - (\sqrt{2} - 1)],$   
 $\bar{y} = \frac{c}{2} [\log_e(1 + \sqrt{2}) - (2 - \sqrt{2})].$  (vertex as origin.)
5.  $\bar{x} = \frac{4a}{3\pi}, \bar{y} = \frac{4b}{3\pi}; \bar{x} = \frac{3}{8}a.$  7.  $\frac{75}{128}$  in. from centre of sphere.
10.  $\frac{2}{3}h$  from base. 11.  $\frac{n+3}{2(n+4)} a$  from centre.
12.  $\frac{\pi h}{3} [r_1^2 + r_1 r_2 + r_2^2].$  17.  $\pi a^2 t; \frac{5}{4}a$  from tangent.
18.  $\bar{x} = 1\frac{19}{81}, \bar{y} = 3\frac{289}{270}; \bar{x} = 1\frac{19}{81}, \bar{y} = \frac{1}{3}.$
19.  $\bar{x} = \frac{1}{2}, \bar{y} = \frac{5}{8}; \bar{x} = \frac{1}{2}, \bar{y} = 2\frac{1}{8}.$
23. Mid-way between the planes.
25.  $\bar{x} = \frac{2a}{3(\pi - 2)}, \bar{y} = \frac{2b}{3(\pi - 2)}.$  26.  $\frac{3a(1 + \cos \alpha)}{8}.$
28. (1.)  $\frac{a}{2},$  (2.)  $\frac{\pi a \sin \alpha}{4a}$  from centre of sphere where  $2\alpha$  is the angle between the planes cutting off the lune.
29. Centre of inscribed circle. 32.  $2km^2(\sqrt{2} - 1).$
33.  $\frac{8kmpx}{(p^2 + x^2)\sqrt{p^2 + 2x^2}}.$  39.  $\left[ \frac{\pi(a^2 + l^2)^{\frac{1}{2}}}{klma} \right]^{\frac{1}{2}}.$
41. 1 hr. 25 min. to nearest min.



## PROBLEMS VI

1.  $\frac{Mg}{\sqrt{1+\mu^2}}(\sin \alpha + \mu \cos \alpha)$  at  $\tan^{-1}\mu$  with the incline.
2.  $\frac{2}{3}\mu W$ .
5. At extremities of axes or at  $x = \pm \frac{ab}{\left[b^2 + a^2\left(\frac{a^2\lambda}{b^2\mu}\right)^{\frac{2}{n-1}}\right]^{\frac{1}{2}}}$ ,  
 $y = \pm \frac{ab}{\left[a^2 + b^2\left(\frac{b^2\mu}{a^2\lambda}\right)^{\frac{2}{n-1}}\right]^{\frac{1}{2}}}$ . When  $n = 1$ , at the extremities of the axes only.
6.  $a\mu^2, \sqrt{\frac{h}{a}}$ .
7.  $a\left[1 - \frac{a}{\sqrt{a^2 + \mu^2 b^2}}\right]$ .
35. Constant  $+ mga[2 \sin \theta + \cos(2\theta \pm \alpha)]$ .

## PROBLEMS VII

1.  $M \frac{d^2(l_1 + 3l_2)}{6(l_1 + l_2)}$ .
2.  $123\frac{5}{8}$  in.<sup>4</sup>
3.  $48\frac{1}{2}$  in.<sup>4</sup>
13.  $\frac{2(m_1 - m_2)g}{2(m_1 + m_2) + M}$ .
15. 420 lb. ft.<sup>3</sup>
16. 1296 rev. per min.
17. (i.)  $265\frac{2}{3}$ ; (ii.)  $106,266\frac{2}{3}$  lb. ft.<sup>3</sup>
18.  $4819.2$  ft.-lb.;  $51.13$  rev.
19.  $\frac{3\sqrt{7}}{10}$ .
21.  $\sqrt{\frac{g}{a}}$ .
22.  $\sqrt{\frac{20ag}{7}}$ ;  $2mga$ .
24.  $\sqrt{\frac{2g}{3a}(\pi \sin \alpha - \cos \alpha)}$ ;  $\frac{g \sin \alpha}{2a}$ ;  

$$\sqrt{\frac{a}{g}} \int_0^{2\pi} \sqrt{\frac{2 - \cos \theta}{2\theta \sin \alpha - \cos \alpha + \cos(\theta + \alpha)}} d\theta.$$
28.  $\frac{2l(M + 3m)}{3(M + 2m)}$ .
29.  $\frac{2(l^2 - 3lx + 6x^2)}{3(l - 4x)}$ .
30. (i.)  $\frac{\sqrt{3}l}{2}$ ; (ii.)  $\frac{5\sqrt{3}l}{12}$ .
31.  $2\pi \sqrt{\frac{(303 + 64n)a}{12(5 + 2n)g}}$ .
33.  $2\pi \sqrt{\frac{a(\cos^2 \alpha + \frac{1}{3} \sin^2 \alpha)}{g \cos \alpha}}$ .



$$31. \pi \sqrt{\frac{7\pi}{g}} \text{ sec}$$

33. On the line through the C.G. at right angles to the line of action of  $J$  and at a distance of  $\frac{a}{3}$  from the C.G. :  $\frac{J}{m}$ ,  $\frac{3J}{ma}$ .

$$36. \frac{ga[m\{(1+\mu^2)\sin\alpha - \mu\} - \mu M]}{(1+\mu^2)(ma^2 + I)\sin\alpha - \mu ma^2}.$$

41. At a distance  $\frac{3}{8}$  of the side of the square from C.

$$42. \frac{1}{2}mu.$$

43.  $\frac{1}{2}\pi nl^2$  ft. from the centre of the rod where  $l$  ft. is the length of the rod and  $n$  is the number of revolutions made by the rod.

### PROBLEMS X

1.  $\frac{3v}{4a}$  where  $a$  is the length of the side of the square.

2.  $\frac{l}{2\sqrt{3}}$  from the centre where  $l$  is the length of the bar.

3.  $\frac{4\pi}{8}(2\sqrt{2} - \sqrt{3} - 1)$  cm.

4. Taking the initial position of the string as the axis of  $x$  and the centre of the string as origin,  $u$  the initial velocity of the particle struck and  $2a$  the length of the string, the paths of the particles are given by  $x = a \cos \frac{ut}{2a}$ ,  $y = \frac{ut}{2} + a \sin \frac{ut}{2a}$ ;  
 $x = -a \cos \frac{ut}{2a}$ ,  $y = \frac{ut}{2} - a \sin \frac{ut}{2a}$  where  $t$  is any time measured from the instant the particle is struck.

$$8. \frac{\Omega}{4}.$$

$$20. \frac{Mmg}{m + 3M \cos^2 \alpha}; \quad \frac{3Mg \cos \alpha}{l(m + 3M \cos^2 \alpha)}.$$

$$21. \frac{Mg \cos \alpha}{1 + 3 \cos^2 \alpha}.$$

### PROBLEMS XI

$$1. p_0 e^{-\frac{g\rho_0 h}{p_0}}$$

$$2. 451.8 \text{ m.m.}$$

$$3. \frac{\sqrt{3}}{8}; \quad \frac{1}{8\mu}.$$

$$4. 580\frac{29}{17} \text{ lb. wt.}, \quad 663\frac{19}{17} \text{ lb. wt.}$$

$$5. 6 \text{ in.}, 4 \text{ in.}$$

$$6. 792w \text{ lb. wt.}, \quad 6\frac{1}{4} \text{ ft. from bottom}; \quad 396w \text{ sec. } 70^\circ \text{ lb. wt.};$$

$$\frac{801w \text{ sec } 70^\circ}{8} \text{ lb. wt. and } \frac{2367w \text{ sec } 70^\circ}{8} \text{ lb. wt. each acting}$$

horizontally and making with the plane of the gate an angle of  $20^\circ$  ( $w$  lb. is the wt. of 1 cub. ft. of water and atmospheric pressure is neglected).

9. 309 tons wt. (nearly) ; 8.4 ft. from the bottom.
10. Greater than 18.276 ft.      11.  $\frac{7\sqrt{2}}{12}a$ .
13.  $\frac{6k^2 + 8kh + 3h^2}{2(3k + 2h)}$  ;  $\frac{2}{3}h$
20. A root of equation  $x^3 - 3ax^2 + \frac{6Fd}{\rho l} = 0$  where  $l$  is the length of the face.
21.  $\frac{2V}{a^2}[x\sqrt{3(3x^2 - a^2)} - (3x^2 - a^2)]$ .
22.  $\frac{wa}{6b}(h^3 - h'^3)$  ;  $\frac{wa}{6b}[3b(h^2 - h'^2) - (h^3 - h'^3)]$ .
23.  $\frac{2\sqrt{gh}}{a}$  where  $h$  is the height and  $a$  the radius of the cylinder.
24.  $\frac{\sqrt{dg}}{r}$  ;  $\frac{\pi r^2 d}{16}$  ;  $\frac{85}{256}\pi r^2 d$ .
25.  $\frac{\pi r^3}{12g}(8g - 3r\omega^2)$  ;  $\frac{\sqrt{2g(r-d)}}{r}$ .

## PROBLEMS XII

1.  $w\pi a^2\sqrt{h^2 + \frac{4}{3}a^2}$ .      7. Unstable ;  $\sec^{-1}\sqrt{2.4}$ .
9.  $\frac{1}{2}[(h + d + 2a) - \sqrt{(h + d)^2 + 4ah}]$
11.  $60^\circ$ .      17.  $\frac{3}{4}a$  ;  $\frac{81}{256}$ .
18.  $2\pi\sqrt{\frac{hs}{g}}$ .      19. 69,528 ft. lb.
21. 83,471 ft. lb.      22. 5.645 lb. per sq. in., 7.542 ft. lb.
23. .723 of its length.      24. 8.23 ft.
26. 9 in. ; 7.542 in.
27.  $x$  ft. below ground level where  $x$  is determined by the equation
- $$\log_e \frac{10}{10-x} = \frac{30+x}{288}.$$
28. 4.9 in.      29. 2 cm. ;  $252\pi g$  ergs.
30. 49 min. 49 sec. ; 11.43 atmospheres.

## PROBLEMS XIII

$$1. s \frac{dv}{ds} = \frac{v(v-u)}{u}.$$

$$2. \sqrt{\frac{2g(a^2 + x^2)(x + 2a - 2\sqrt{a^2 + x^2})}{a^2 + 3x^2}},$$

$$g \left[ \frac{\sqrt{a^2 + x^2}(\sqrt{a^2 + x^2} - 2x)}{a^2 + 3x^2} - \frac{4a^2x(x + 2a - 2\sqrt{a^2 + x^2})}{(a^2 + 3x^2)^2} \right]$$

where  $AB = 2a$ .

$$5. \sqrt{2k \left( \frac{1}{d} - \frac{2}{\sqrt{4d^2 + a^2}} \right)}; \frac{1}{2} a \sqrt{\mu}.$$

$$12. 11.94 \text{ ft./sec.}; 19.52 \text{ sec.}$$

$$13. 37.04 \text{ ft.}; 3.043 \text{ sec.}; 47.43 \text{ ft./sec.}$$

$$14. 1.267 \text{ sec.}$$

$$16. \frac{4\pi}{3\mu}; \frac{3\sqrt{3}g}{\mu^2}.$$

$$19. \frac{1}{368} [\sqrt{2} \sin 8\sqrt{2} \cdot t - .4 \sin 40t].$$

$$22. .0169.$$

## PROBLEMS XIV

$$1. ue^{-kx}; \frac{g}{k} (1 - e^{-kx}).$$

$$3. y = a \cosh \frac{x}{a}, \text{ where } a \text{ is the distance from the line (axis of } x) \text{ when the velocity perpendicular to it is zero.}$$

$$6. x = u \sqrt{\frac{a}{2k}} \left[ a \sin^{-1} \sqrt{\frac{y}{a}} + \sqrt{y(a-y)} \right].$$

$$14. 1^\circ 1'.$$

$$16. 2 \tan^{-1} \frac{\mu}{pV^2}.$$

$$18. 27.12 \text{ days.}$$

$$31. a(1 - \sin a) \text{ where } a \text{ is a root between } 0 \text{ and } \pi \text{ of}$$

$$e^{2\mu\theta} \left[ 3\mu \cos \theta + (1 - 2\mu^2) \sin \theta \right] = 3\mu.$$

## PROBLEMS XV

$$1. \frac{2}{\sqrt{2}} \text{ of the length of the beam from the supported end.}$$

$$3. \frac{\sqrt{596 - 30\sqrt{3}}}{8} \text{ tons wt.}; \frac{17 + \sqrt{3}}{8} \text{ tons wt.}; \tan^{-1} \frac{19 + 18\sqrt{3}}{52};$$

$$\frac{15(13 + \sqrt{3})}{16} \text{ tons wt. ft.}$$

4.  $\frac{\sqrt{2}-1}{2}$  of the length.      5.  $\frac{1}{3}$  of the length from the free end.  
 7.  $58\frac{1}{2}$  tons wt. ft.      10.  $2.17 \times 10^8$  gm. wt. per sq. cm.  
 12.  $3.18 \times 10^7$  lb. wt. per sq. in.  
 13. 16.3 ft.      14. 12.02 sec.  
 16. 6 in.      17. .207 in.  
 18.  $1\frac{1}{4}\frac{2}{3}$  in      19. 9.6 in.  
 20. .0060 in      22. 30.43 ft.

## PROBLEMS XVI

1.  $\mathbf{b} - \mathbf{a}, -\mathbf{a}, -\mathbf{b}, \mathbf{a} - \mathbf{b}$ .  
 4.  $\sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]}$ , etc.  
 7.  $-1\mathbf{i}; 4; -11\mathbf{i} + 13\mathbf{j} + 17\mathbf{k}$ .  
 10.  $15\mathbf{i} + \mathbf{j} + 18\mathbf{k}$ .

## APPENDIX I

1.  $\frac{14}{9} Ma^2, \frac{10}{3} Ma^2, \frac{17}{9} Ma^2, \frac{1}{2} \tan^{-1} \left( \frac{17}{8} \right)$ .

## MISCELLANEOUS PROBLEMS

6.  $2\frac{1}{4}$  ft. lb.      8. 187.036 Kg. wt.  
 9.  $2\sqrt{6}$  in., 7 in.  
 10.  $\frac{1}{2}(8 + 2x - x^2)$  lb. wt.;  $\frac{x}{2}\sqrt{g}$  ft./sec.  
 16.  $\frac{m(u-u)'}{M}, -\frac{mv'}{M}$ .  
 21.  $c = \frac{abmg}{\lambda(a+b)}$ ; not less than  $\frac{2amg}{a+b}$ .  
 26.  $\frac{mu^2}{2a}$ ; Moves in an ellipse with constant velocity  $u$ ;  $\frac{mau^2}{2(a^2-l^2)}$ .  
 33.  $\frac{2l}{3} \left( 1 \pm \frac{1}{n} \right)$  from A;  $\frac{2l}{3}$  from A.  
 37.  $\frac{P}{2}$  in the direction of P;  $\frac{3P}{2lm}$ ;  $\frac{3P^2}{2\mu g l m^2}$ .  
 42. 687.2 days.      43.  $\frac{1}{3}$  of K.E. lost.  
 46.  $\frac{\mu mg}{\sqrt{1+\mu^2}}; \frac{mg(\mu \cos a - \sin a)}{\sqrt{1+\mu^2}}$  making angle of  $\tan^{-1} \mu$  with the plane.  
 56.  $HM = \frac{1}{3}r$ , unstable.  
 58.  $\lambda\sqrt{65}, 4\gamma = 7x + 44$ .



